- For each permutation π of k items, $P_{i_1,\ldots,i_k}(A) = P_{i_{\pi(1)},\ldots,i_{\pi(k)}}(B)$, where
$$B = \{(x_{\pi(1)},\ldots,x_{\pi(k)}) : (x_1,\ldots,x_k) \in A\}.$$
- For each $\ell \in R \setminus \{i_1,\ldots,i_k\}$, $P_{i_1,\ldots,i_k}(A) = P_{i_1,\ldots,i_k,\ell}(B)$, where
$$B = \{(x_1,\ldots,x_k,x_{k+1}) : (x_1,\ldots,x_k) \in A, x_{k+1} \in \mathcal{X}_\ell\}.$$

Since the set R may not be ordered, the first condition ensures that it does not matter in what order one writes a finite set of indices. The second condition is the substantive one, and it ensures that the marginal distributions of subsets of coordinates are the probability measures associated with those subsets.

To avoid excessive notation, it will be convenient to refer to P_J as the probability measure associated with a finite subset $J \subseteq R$ without specifying the order of the elements of J. When the consistency conditions in Definition B.132 hold, this should not cause any confusion.

The proof of the following theorem is adapted from Loève (1977, pp. 94–5). The theorem says that consistent finite-dimensional distributions determine a unique joint distribution on the product space.

Theorem B.133.[61] *Let $\mathcal{X} = \prod_{r \in R} \mathcal{X}_r$ with the product σ-field, where $\mathcal{X}_r$ is a Borel space for every r. For each finite subset $J \subseteq R$, let P_J be a probability measure on $\prod_{r\in J} \mathcal{X}_r$. Suppose that the P_J are consistent as defined in Definition B.132. Then there exists a unique distribution on $\mathcal{X}$ with finite-dimensional marginals given by the P_J.*

Proof. The uniqueness follows from Theorem B.131, if we can prove existence. First, suppose that $\mathcal{X}_r = \mathbb{R}$ for all r. Let $\mathcal{C}$ be the class of all unions of finitely many finite-dimensional cylinder sets of the form $C = \prod_{r\in R} C_r$, where all but finitely many of the C_r equal $\mathbb{R}$ and the others are unions of finitely many intervals. The class $\mathcal{C}$ is a field. For C of the above form, define $P(C) = P_J(\prod_{r\in J} C_r)$. The consistency assumption implies that P can be uniquely extended to a finitely additive probability on $\mathcal{C}$. To show that P is countably additive, we will show that if $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of elements of $\mathcal{C}$ such that $P(A_n) > \epsilon$ for all n, then $A = \cap_{n=1}^\infty A_n$ is nonempty. Suppose that $P(A_n) > \epsilon$ for all n. Let J_n be the set of all subscripts involved in $A_1,\ldots,A_n$ and J be the union of these sets. Let $A_n = B_n \times \prod_{r\notin J_n} \mathcal{X}_r$. Then $P(A_n) = P_{J_n}(B_n)$, and B_n is the union of finitely many products of intervals. For each product of intervals H that constitutes B_n, we can find a product of bounded closed intervals contained in H such that the P_{J_n} probability of the union of these H is as close as we wish to $P_{J_n}(B_n)$. Let C_n be a finite union of products of closed bounded intervals contained in B_n such that $P_{J_n}(B_n \setminus C_n) < \epsilon/2^{n+1}$. If D_n is the cylinder set corresponding to C_n, then
$$P_{J_n}(A_n \setminus D_n) = P_{J_n}(B_n \setminus C_n) < \frac{\epsilon}{2^{n+1}}.$$
Now, let $E_n = A_n \cap_{i=1}^n D_i$, so that $P(A_n \setminus E_n) < \epsilon/2$. It follows that $P(E_n) > \epsilon/2$, so each E_n is nonempty. Let $x^n = (x_1^n, x_2^n, \ldots) \in E_n$. Since $E_1 \supseteq E_2 \supseteq \cdots$, it follows that for every $k \geq 0$, $x^{n+k} \in E_n \subseteq D_n$. Hence $(x_i^{n+k}; i \in J_n) \in C_n$. Since

[61] This theorem is used in the proof of Lemma 2.123.