quantity, and that the agent has implicitly agreed to accept all gambles of the form $c(Y-\mu)$ for $-1 \le c \le 1$, where $\mu = \sum_{x=1}^{\infty} c_x 2^{-x}$. If the agent were foolish enough to be indifferent between all gambles of the form $d(Y-p)$ for $-a \le d \le a$ where $p \neq \mu$, then a clever opponent could make money with no risk. For example, if $p > \mu$, let $f = \min\{1, a\}$. The opponent would ask the agent to accept the gamble $f(Y-p)$ as well as the gambles $-fc_x(I_x - 2^{-x})$ for $x = 1, 2, \ldots$. The net effect to the agent of these gambles is $-f(p-\mu) < 0$, no matter what value X takes! A similar situation arises if $p < \mu$. Only $p = \mu$ protects the agent from this sort of problem, which is known as *Dutch book*.

To avoid Dutch book, we introduce the following definition.

Definition B.137. Let $\{X_\alpha : \alpha \in \mathrm{A}\}$ be a collection of bounded random variables. Suppose that, for each α, an agent gives a prevision $P(X_\alpha)$ and is indifferent between all gambles of the form $c(X_\alpha - P(X_\alpha))$ for $-d_\alpha \le c \le d_\alpha$ with $d_\alpha = \min\{\frac{M}{\max X_\alpha - P(X_\alpha)}, \frac{M}{P(X_\alpha) - \min X_\alpha}\}$ for some $M > 0$. These previsions are *coherent* if there exist no countable subset $B \subseteq \mathrm{A}$ and $\{c_b : -d_b \le c_b \le d_b, \text{ for all } b \in B\}$ such that $-M \le \sum_{b \in B} c_b(X_b - P(X_b)) < 0$ under all circumstances.[65] If a collection of previsions is not coherent, we say that it is *incoherent.*

The value M is the maximum amount the agent is willing to lose. Coherence of a sufficiently rich collection of previsions is equivalent to a probability assignment.

Theorem B.138.[66] *Let $(S, \mathcal{A})$ be a measurable space. Suppose that, for each $C \in \mathcal{A}$, the agent assigns a prevision $P(I_C)$, where I_C is the indicator of C. Define $\mu : \mathcal{A} \to I\!R$ by $\mu(C) = P(I_C)$. Then the previsions are coherent if and only if μ is a probability on $(S, \mathcal{A})$.*

Proof. Without loss of generality, suppose that the agent is indifferent between all gambles of the form $c(I_C - P(I_C))$, for all $-1 \le c \le 1$. For the "if" part, assume that μ is a probability. Let $\{C_n\}_{n=1}^{\infty} \in \mathcal{A}$ and $c_i \in [-1, 1]$ be such that with

$$X = \sum_{n=1}^{\infty} c_n(I_{C_n} - \mu(C_n)),$$

the maximum losses from X and from $-X$ are small enough for the agent to afford. Since this makes X bounded, it follows from Fubini's theorem A.70 that $\mathrm{E}(X) = 0$; hence it is impossible that $X < 0$ under all circumstances, and the previsions are coherent.

For the "only if" part, assume that the previsions are coherent. Clearly, $\mu(\emptyset) = 0$, since $I_\emptyset = 0$ and $-c\mu(\emptyset) \ge 0$ for both positive and negative c. It is also easy to see that $\mu(A) \ge 0$ for all A. If $\mu(A) < 0$, then for all negative c, $c(I_A - \mu(A)) < 0$ and we have incoherence. Countable additivity follows in a similar fashion. Let $\{A_n\}_{n=1}^{\infty}$ be mutually disjoint, and let $A = \cup_{n=1}^{\infty} A_n$. If $\mu(A) < \sum_{n=1}^{\infty} \mu(A_n)$,

[65] When only finitely many gambles are required to be combined at once, as by DeFinetti (1974), incoherence requires that the sum be strictly less than some negative number under all circumstances. That is, DeFinetti would allow a strictly negative gamble to be called coherent, so long as the least upper bound was 0.

[66] This theorem is used in the proof of Theorem B.139.