Homework 1 (due Thursday May 23rd at 11:59pm)

1. (10 pts - medium - De Morgan's laws) Prove that

$$(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c$$

where $A_1, \ldots, A_n$ are subsets of the universe set Ω. Your proof should hold also in the case of n not being a finite number (e.g. you can not prove this by induction).

Solution: Consider $x \in (\cup_{i=1}^{n} A_i)^c$. Then the following equivalences hold

$$\begin{aligned} x \in (\cup_{i=1}^{n} A_i)^c &\iff \nexists \text{i s.t. } x \in A_i \\ &\iff \forall i \; x \in A_i^c \iff x \in \cap_{i=1}^{n} A_i^c. \end{aligned}$$

2. (20 pts - easy) Imagine that you throw a die (6 faces). The two events of interest are:

- (A) you obtain an even number;
- (B) you obtain a number larger than 3.

Work on the following tasks:

- write down the sample space Ω;
- prove/disprove that A and B are disjoint;
- compute the probability of $A \cap B$;
- compute the probability of $A \cap B^c$;
- for an outcome x, compute the probability of $x \in B$ given that $x \in A$.

Solution:

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- The events are not disjoint. Indeed, if we choose $x = 4$ then we have $x \in A \cap B$.
- $P(A \cap B) = P(\{4, 6\}) = 1/3$.
- $P(A \cap B^c) = P(\{2\}) = 1/6$.
- $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = 2/3.$$

3. (10 pts - easy) Let A and B be sets in the universe set Ω. Let also P be a probability measure on Ω. Prove that

   - $A = (A \cap B) \cup (A \cap B^c)$;
   - $P(A) = 1 - P(A^c)$;
   - $P(A) = P(A \cap B) + P(A \cap B^c)$;

   Solution:

   - $A = A \cup \Omega = A \cup (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$;
   - $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c) \iff P(A) = 1 - P(A^c)$;
   - $P(A) = P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c) - P(\emptyset) = P(A \cap B) + P(A \cap B^c)$.

4. (20 pts - medium) Suppose that $A \subset B \subset \Omega$. Prove that

   - $B^c \subset A^c$;
   - $P(B \setminus A) = P(B) - P(A)$.

   Solution:

   - Suppose that $x \in B^c$, but $x \notin A^c$. Then $x \in A$, and the assumption $A \subset B$ this implies $x \in B$ as well. This is a contradiction, and we conclude that $B^c \subseteq A^c$. To prove that $B^c \subset A^c$ notice that since $A \subset B$, there must be some element z such that $z \in B$ but $z \notin A$. Therefore $z \in A^c$ but $z \notin B^c$.This concludes the argument.
   - From the third axiom we know that

     $$P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B) = P(A) + P(B \setminus A).$$

     Rearranging the terms we get

     $$P(A \setminus B) = P(B) - P(A).$$

5. (10 pts - easy) Write down your name. How many different words can you form with those letters? In order to simplify things, consider only only words whose length matches the length matches the length of your name.

   Solution: My name is Riccardo, hence I have 2 "r", 2 "c", 1 "i", 1 "a", 1 "d", 1 "o". There are 8!/4 possible words with these letters.

6. (20 pts - harder) Assuming the probabilities of having a birthday in each month are equally likely,

   - (pigeonhole principle) compute the probability that none of them celebrates the birthday in the same month given that there are 13 people;
   - (birthday problem) compute the probability that none of them celebrates the birthday in the same month given that there are 12 people;
   - compute the probability that at least two of them celebrate their birthday in the same month given that there are 12 people;
   - choose a random person. What is the probability that the remaining 11 people all have the same birthday as this random person?

   Solution:

   - It is zero!
   - There are 12^{12} possible outcomes for the birthdays. If nobody can have the same month, then we also have P_{12} permutations with all different months, that is 12!. Hence the result is
   $$\frac{12!}{12^{12}}$$
   - This is given by
   $$1 - \frac{12!}{12^{12}}.$$
   - This is given by
   $$\frac{1}{12^{11}}.$$

7. (10 pts - medium) You are given a fair die (6 faces). You throw it $n-$times. Compute the probability that the sum of the faces is

   - n;
   - $n+1$;
   - $n+2$.

   Solution:

   - This is simply 6^{-n} since every outcome needs to be 1.

- This is given by

$$\frac{\binom{n}{1}}{6^n}.$$

- This is given by

$$\frac{\binom{n}{2}}{6^n} + \frac{\binom{n}{1}}{6^n}.$$