Homework 10 (due Thursday, June 27 at 11:59pm)

1. (20 pts) Consider a *random walk* process on the integers $\mathbb{X} = \{X_i\}_{i=0}^{\infty}$ with $P(X_0 = x_0) = 1$ for some $x_0 \in \mathbb{Z}$. The process has state space $\mathcal{S} = \mathbb{Z}$, the set of integer numbers. Furthermore, at any time $i \in \{0, 1, \dots\}$ we have

$$P(X_{i+1} = x_i + 1 | X_i = x_i) = p$$
$$P(X_{i+1} = x_i - 1 | X_i = x_i) = 1 - p.$$

- Does the random process $\mathbb{X}$ satisfy the Markov property?
- Consider the new process $\mathbb{Y} = \{Y_i\}_{i=0}^{\infty}$ with $Y_i = \sum_{j \le i} X_j$. Does the random process $\mathbb{Y}$ satisfy the Markov property?

Solution:

- Yes, $\mathbb{X}$ satisfies the Markov property: for any $i \in \{0, 1, \dots\}$, the state of the process $\mathbb{X}$ at time $i+1$ only depends on the state of the process at time i.
- Let $Y_i = \sum_{j \le i} X_i$. Notice that $Y_{i+1} = Y_i + X_{i+1}$. We have

$$\begin{aligned}
&P(Y_{i+1} = y_{i+1} | Y_i = y_i, Y_{i-1} = y_{i-1}, \dots, Y_0 = y_0) \\
&= P(Y_i + X_{i+1} = y_{i+1} | Y_i = y_i, Y_{i-1} = y_{i-1}, \dots, Y_0 = y_0) \\
&= P(y_i + X_{i+1} = y_{i+1} | Y_i = y_i, Y_{i-1} = y_{i-1}, \dots, Y_0 = y_0) \\
&= P(X_{i+1} = y_{i+1} - y_i | Y_i = y_i, Y_{i-1} = y_{i-1}) \\
&= P(X_{i+1} = y_{i+1} - y_i | X_i = y_i - y_{i-1}) \\
&= \begin{cases} p \text{ if } y_{i+1} - y_i = y_i - y_{i-1} + 1 \\ 1 - p \text{ if } y_{i+1} - y_i = y_i - y_{i-1} - 1 \end{cases} \\
&= \begin{cases} p \text{ if } y_{i+1} = 1 + 2y_i - y_{i-1} \\ 1 - p \text{ if } y_{i+1} = -1 + 2y_i - y_{i-1} \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&P(Y_{i+1} = y_{i+1} | Y_i = y_i, Y_{i-1} = y_{i-1}, \dots, Y_0 = y_0) \\
&= P(Y_{i+1} = y_{i+1} | Y_i = y_i, Y_{i-1} = y_{i-1})
\end{aligned}$$

and $\mathbb{Y}$ does not satisfy the Markov property.

2. (20 pts) For a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ .3 & .1 & .1 & .1 & 0 & .2 & .2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & .1 & 0 & .2 & 0 & .4 \\ .2 & .4 & 0 & .3 & 0 & 0 & .1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Classify the states (recurrent and transient). Does the chain admit a limiting distribution?

Solution:
The chains contains 3 recurrent classes: $\{a\}$, $\{b, g\}$ and $\{d\}$. The others are transient. This chain does not admit a limiting distribution because there are more than one recurrent class (i.e. the limiting distribution depends on the initial state).

3. (20 pts) Let $X_0, X_1, \cdots$ be a Markov Chain with state space $\{1, 2, 3\}$, initial distribution $p_{X_0} = (1/5, ?, 2/5)$ and transition probability matrix

$$P = \begin{bmatrix} 1/5 & 4/5 & ? \\ 2/5 & 1/2 & ? \\ 0 & 1/10 & ? \end{bmatrix}$$

Fill in the entries for P and p_{X_0}, and answer the following:

(a) Compute $P(X_1 = 1|X_0 = 2)$.

(b) The row vector p_{X_0} describes the distribution of X_0 . What is the row vector describing the distribution of X_1 ?

(c) What is $P(X_1 = 3)$?

(d) What is the row vector describing the distribution of X_2 ?

(e) What is $P(X_2 = 1)$?

Solution:
The probabilities along the rows of the transition matrix must sum to 1, as must the initial distribution. Hence we have

$p_{X_0} = (1/5, 2/5, 2/5)$,

and

$$P = \begin{bmatrix} 1/5 & 4/5 & 0 \\ 2/5 & 1/2 & 1/10 \\ 0 & 1/10 & 9/10 \end{bmatrix}$$

(a) (3 points) $P(X_1 = 1|X_0 = 2) = P_{21} = 2/5 = .4$

(b) (3 points) The row vector describing X_1 is obtained by matrix multiplication:

$$p_{X_1} = p_{X_0} * P = (.2, .4, .4) \begin{bmatrix} .2 & .8 & 0 \\ .4 & .5 & .1 \\ 0 & .1 & .9 \end{bmatrix} = (.2, .4, .4)$$

Note that this matrix multiplication implements the law of total probability for each value that X_1 could take, namely:

$$P(X_1 = k) = \sum_{i=1}^{3} P(X_1 = k|X_0 = i)P(X_0 = i)$$

(c) (3 points) $P(X_1 = 3) = p_{X_1}[3] = .4$ using the distribution calculated in the previous part.

(d) (3 points) To get the row vector describing X_2 we need to use the 2-step transition matrix P_2 which is just the square of P:

$$P_2 = P * P = \begin{bmatrix} .2 & .8 & 0 \\ .4 & .5 & .1 \\ 0 & .1 & .9 \end{bmatrix} \begin{bmatrix} .2 & .8 & 0 \\ .4 & .5 & .1 \\ 0 & .1 & .9 \end{bmatrix} = \begin{bmatrix} .36 & .56 & .08 \\ .28 & .58 & .14 \\ .04 & .14 & .82 \end{bmatrix}$$

Now we compute the probability distribution for X_2 using the law of total probability again:

$$P(X_2 = k) = \sum_{i=1}^{3} P(X_2 = k|X_0 = i)P(X_0 = i).$$

Or in matrix form:

$$p_{X_2} = p_{X_0} * P_2 = (.2, .4, .4) \begin{bmatrix} .36 & .56 & .08 \\ .28 & .58 & .14 \\ .04 & .14 & .82 \end{bmatrix} = (.2, .4, .4)$$

(e) (3 points) $P(X_2 = 1) = p_{X_2}[1] = .2$ using the distribution calculated in the previous part.

4. (20 pts) The *Ehrenfest* model for heat exchange can be described as follows: two urns, A and B, contain a total of $2N$ balls. At every step, each ball is equally likely to be drawn among all the $2N$ balls. It is then put into the other urn. Let X_n be the number of balls in urn A immediately after the n-th step.

   (a) Show that $X_0, X_1, X_2, \ldots$ is a Markov Chain and find the transition probability matrix.

   (b) Let $\mu_{i,n} = E[X_n | X_0 = i]$, $i = 0, \ldots, 2N$. Show that
   $$\mu_{i,n} = 1 + (1 - 1/N)\mu_{i,n-1}, \quad \text{for all } n > 0.$$

   (c) Show that
   $$\mu_{i,n} = N + (i - N)(1 - 1/N)^n$$

   (d) Find $\lim_{n\to\infty} E[X_n]$.

   Solution:

   (a) The number of balls in urn A on next state depends only on that on the current state, and X_n is defined for $0, 1, .., N$. Thus, X_n is a Markov Chain.

   Let the transition probability matrix be $P = p_{ij}$, where p_{ij} is the probability from state i to state j for time $t - 1$ to time t. Given that the process is in state i at time $t - 1$, the probability that the selected random integer corresponds to a number on a ball in urn A is $\frac{i}{2N}$. Then, there will be $j = i - 1$ balls in the urn A at time t. Similarly, the probability that the selected integer corresponds to a number on a ball in urn B is $\frac{2N-i}{2N} = 1 - \frac{i}{2N}$ at time t-1, in which case there will be $j = i + 1$ balls in the urn A at time t. Thus, if $0 < i < N$
   $$p_{ij} = \begin{cases} \frac{i}{2N},\ for\ j = i - 1 \\ 1 - \frac{i}{2N},\ for\ j = i + 1 \\ 0,\ otherwise \end{cases}$$

   and $p_{0,0} = 1$, $p_{0,1} = 1$, $p_{2N,2N} = 0$ and $p_{2N,2N-1} = 1$

(b) Notice that

$$E[X_n|X_{n-1}=j] = (j-1)\frac{j}{2N}+(j+1)\left(1-\frac{j}{2N}\right) = 1+j\left(1-\frac{1}{N}\right)$$

Then

$$\begin{aligned} E[X_n|X_0=i] &= \textstyle\sum_{j\in S} E[X_n|X_{n-1}=j, X_0=1]P(X_{n-1}=j|X_0=i) \\ &= \textstyle\sum_{j\in S}\left(1+j\left(1-\frac{1}{N}\right)\right)P(X_{n-1}=j|X_0=i) \\ &= 1+\left(1-\frac{1}{N}\right)\mu_{i,n-1} \end{aligned}$$

Writing $X_n = X_{n-1}+(X_n-X_{n-1})$ and conditioning this equation on X_{n-1} and X_0, we obtain

$$\mu_{i,n} = 1+(1-\frac{1}{N})\mu_{i,n-1}, \quad \text{for all } n>0.$$

(c) Using induction, we iterate part (b) to obtain

$$\mu_{i,n} = N+(1-N)(1-\frac{1}{N})^n$$

(d)

$$lim_{n\to\infty}E[X_n] = N$$

5. (20 pts) Suppose a student in 36-217 can either be up-to-date (U) with them material covered in class or behind (B). The probability of a student being up-to-date or behind on a particular week depends on whether he/she has been behind or up-do-date in the previous two weeks. In particular

- If behind both this week and last week, the student will be behind next week as well with probability 0.8
- If up-to-date both this week and last week, the student will be up-to-date next week as well with probability 0.9.
- If behind last week and up-to-date this week, the student will be behind with probability 0.5.
- If up-to-date last week and behind this week, the student will be behind with probability 0.7.

(a) Is this a first-order Markov chain? Why?

(b) Explain how you can enlarge the state space and obtain a first-order Markov chain.

(c) More generally, if you have a k-th order Markov chain on a state space of cardinality m, explain how you can always derive a first-order Markov chain on a larger state space and find the cardinality if this enlarged state space.

(a) (2 points) This is not a first-order Markov chain since the probability of being in a given state depends on the states of the two previous steps.

(b) (3 points) We can consider the state space $\{U, B\}^2 = \{(U, U), (U, B), (B, U), (B, B)\}$ which is the set of paths that matter for determining the states of the next step. We consider the new random variable $Y_t = (X_t, X_{t-1})$. In order to define the probabilities of the new first-order Markov chain we define

$$P(Y_t = (X_t, X_{t-1}) = (i, j)|Y_{t-1} = (X_{t-1}, X_{t-2}) = (k, l)) = \begin{cases} P(X_t = i|X_{t-1} = k, X_{t-2} = l) \\ 0 \quad \text{if} \quad j \neq k \end{cases}$$

Thus, in this case we obtain the probability transition matrix for the states $\{(U, U), (U, B), (B, U), (B, B)\}$

$$\begin{pmatrix} 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(c) (4 points) In general if we have a state space $\{E_1, \ldots E_m\}$ we can consider the new state space $\{E_1, \ldots E_m\}^k$ with cardinality m^k which contents the set of paths that matter for determining the states of the next step. We consider the new random variable $Y_t = (X_t, X_{t-1}, \ldots, X_{t-(k-1)})$. In order to define the probabilities of the new first-order Markov chain we define

$$P(Y_t = (X_t, X_{t-1}, \ldots, X_{t-(k-1)}) = (i_0, i_1, \ldots, i_{k-1})|Y_{t-1} = (X_{t-1}, X_{t-2}, \ldots, X_{t-k}) = (j_1, j_2,$$

$$= \begin{cases} P(X_t = i_0|X_{t-1} = j_1, X_{t-2} = j_2, \ldots, X_{t-k} = j_k) \quad \text{if} \quad i_1 = j_1, \ldots, i_{k-1} = j_{k-1} \\ 0 \quad \text{otherwise} \end{cases}$$