Homework 2 (due Tuesday, May 28 at 11:59pm)

1. (5 pts - independence) Consider two independent events A and B in the universe set Ω.

   - prove/disprove that A^c and B are independent.
   - prove/disprove that A^c and B^c are independent.

   Solution:

   - The events are independent. Indeed,

     $$P(A^c)P(B) = (1-P(A))P(B) = P(B)-P(A)P(B) = P(A^c \cap B).$$

   - Again, independence holds.

     $$\begin{aligned} &P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) \\ &= 1 + P(A)P(B) - P(A) - P(B) \\ &= 1 + P(A \cap B) - P(A) - P(B) \\ &= 1 - P(A \cup B) = P(A^c \cap B^c). \end{aligned}$$

2. (5 pts - axioms) Show that for $A, B \subset \Omega$, $P(A \cap B) \geq \max\{0, P(A) + P(B) - 1\}$.

   Solution:
   Because the probability of any event is a non-negative number, it is clear that $P(A \cap B) \geq 0$. Furthermore, recall that in general for two events A and B we have

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

   from which it follows that

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

   Since $P(A \cup B) \leq 1$, we have

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1.$$

   Thus, since $P(A \cap B) \geq 0$ and $P(A \cap B) \geq P(A) + P(B) - 1$, we conclude that $P(A \cap B) \geq \max\{0, P(A) + P(B) - 1\}$.

3. (5 pts - independence) Let $A, B \subset \Omega$ be such that $P(A) > 0$ and $P(B) > 0$. Prove or disprove the following claim: in general, it is true that $P(A|B) = P(B|A)$. If you believe that the claim is true, clearly explain your reasoning. If you believe that the claim is false in general, can you find an additional condition under which the claim is true?

   Solution:
   The claim is false in general. A simple counter example: take A arbitrary with $0 < P(A) < 1$ and $B = \Omega$. We have $P(A|B) = P(A|\Omega) = P(A)$ while $P(B|A) = P(\Omega|A) = 1$. Notice, however, that if $P(A) = P(B)$, then

   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A).$$

   Therefore, under this additional condition, the claim is true.

4. (5 pts - independence) Let $A, B \subset \Omega$ be such that $P(A) > 0$ and $P(B) > 0$. Show that

   - if A and B are disjoint, then they are not independent
   - if A and B are independent, then they are not disjoint.

   Solution:

   - Suppose that A and B are independent. Then, $P(A \cap B) = P(A)P(B) > 0$ (since $P(A) > 0$ and $P(B) > 0$ by assumption). However, A and B are disjoint by assumption, i.e. $A \cap B = \emptyset$ which implies that $P(A \cap B) = 0$, and we reach a contradiction.
   - Suppose that A and B are disjoint, i.e. $A \cap B = \emptyset$. Then $P(A \cap B) = 0$. However, A and B are independent by assumption, i.e. $P(A \cap B) = P(A)P(B) > 0$ (recall that $P(A) > 0$ and $P(B) > 0$), and again we reach a contradiction.

5. (10 pts - distribution) Find $c \in \mathbb{R}$ such that it makes

   $$f(x) = \frac{2x\mathbb{1}_{[0,1]}(x) + 2(2-x)\mathbb{1}_{(1,2]}(x)}{c}$$

   a valid probability density function.
   Solution:
   We are going to use the fact that $\int f(x)dx = 1$ in order for f to be a

proper pdf.
Let us rewrite $f(x)$ as

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)dx &= \int_{-\infty}^{+\infty} \frac{2x\mathbb{1}_{[0,1]}(x) + 2(2-x)\mathbb{1}_{(1,2]}(x)}{c}dx \\ &= \int_{-\infty}^{+\infty} \frac{2x\mathbb{1}_{[0,1]}(x)}{c}dx + \int_{-\infty}^{+\infty} \frac{2(2-x)\mathbb{1}_{(1,2]}(x)}{c}dx \\ &= \int_0^1 \frac{2x}{c}dx + \int_1^2 \frac{2(2-x)}{c}dx \\ &= \frac{2}{c}\end{aligned}$$

Therefore we need

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \iff c = 2.$$

6. (10 pts - expectation) Let $\alpha > 0$. The pdf of the r.v. X is given by

$$f(x) = \alpha x^{-\alpha-1}\mathbb{1}_{[1,+\infty)}(x).$$

Compute $E[X]$ for

- $\alpha > 1$;
- $0 < \alpha \leq 1$.

What do you notice?

Solution:
This is the Pareto distribution with shape α.

$$E[X] = \int xf(x)dx = \int_1^{+\infty} x\alpha x^{-\alpha-1}dx = \alpha \frac{x^{-\alpha+1}}{1-\alpha}\Big|_1^{+\infty}$$

Now we can easily notice that

- in case of $\alpha > 1$, $E[X] = \alpha/(\alpha - 1)$;
- in case of $0 < \alpha \leq 1$, $E[X] = +\infty$.

7. (10 pts - distribution) We are going to obtain the pdf from the cumulative distribution function. As we have seen in class, the cumulative

distribution function is defined as $F(x) := P(X \leq x)$. For a continuous random variable X, we have

$$f(x) = \frac{\partial F(x)}{\partial x}$$

The cdf we will consider is $F(x) = [1 - e^{-\lambda x}]\mathbb{1}_{[0,\infty)}(x)$. Compute

- $f(x)$. *Hint:* remember that the support of X won't change even if you take the derivative.
- prove that f s a valid pdf.
- $E[X]$.
- $V(X)$.

Solution:
This is the *exponential* distribution.

- Taking the derivative of the pdf, we obtain

$$\frac{\partial[1 - e^{-\lambda x}]\mathbb{1}_{[0,\infty)}(x)}{\partial x} = \lambda e^{-\lambda x}\mathbb{1}_{[0,\infty)}(x)$$

- First, notice that $f(x) \geq 0$. Moreover,

$$\lambda \int_0^{+\infty} e^{-\lambda x} dx = -\lambda e^{-\lambda x}\Big|_0^{+\infty} = 1$$

- Through some integration, $E[X] = \lambda^{-1}$.
- Similarly, $V(X) = \lambda^{-2}$.

8. (15 pts - expectation) Consider the r.v. X with pdf $f(x) = \mathbb{1}_{[0,1]}(x)$.

- Compute $E[X]$.
- Let $A = [0, 1/3]$. Compute $P(X \in A)$.
- let $A = \{0.01, 0.02, 0.03, \ldots, 0.99\}$. Compute $P(X \in A)$. (yes, this again)

Let's suppose that you play a game: you win 100$ if $X \in A$, you lose 100$ if $X \notin A$. Write down the probability mass function for Y, the amount of money you would end up with if you played the game. Would you play the game if

- $A = [0, 1/3]$?
- $A^c = \{0.01, 0.02, 0.03, \ldots, 0.99\}$?

Solution:
This is called the *uniform distribution*. Let's start:

- $E[X] = 1/2$.
- $P(X \in A) = 1/3$.
- $P(X \in A) = 0$.

The pmf is given by

$$p(x) = \begin{cases} P(A) \text{ if } X \in A \\ 1 - P(A) \text{ o/w} \end{cases}$$

- I wouldn't play. Indeed, $E[Y] = 1/3 \cdot 100 - 2/3 \cdot 100 = -1/3 \cdot 100$.
- Of course. $E[Y] = 100$!

9. (10 pts - inequalities) For two r.v.'s X_1 and X_2, prove that $P(X_1+X_2 \geq 0) \leq P(X_1 \geq 0) + P(X_2 \geq 0)$.
Hint: recall the union bound.

Solution:
Notice that $\{X_1 + X_2 \geq 0\} \subset \{X_1 \geq 0\} \cup \{X_2 \geq 0\}$. Then we have

$$P(X_1 + X_2 \geq 0) \leq P(\{X_1 \geq 0\} \cup \{X_2 \geq 0\}) \leq P(X_1 \geq 0) + P(X_2 \geq 0).$$

10. (10 pts - inequalities) Given a continuous positive r.v. X with expectation $\mu := E[X] = 5$, find a lower bound for $P(X < 5)$. Can you also find an upper bound for $P(X \leq 5)$? **Solution:**

$$P(X < 5) = 1 - P(X \geq 5) \geq 1 - \frac{E[X]}{5}$$

by Markov's inequality.
For the upper bound, we can only conclude that $P(X \leq 5) \leq 1$.

11. (15 pts - inequalities) Given a continuous positive r.v. X with expectation $\mu := E[X] = 10$, find an upper bound for

- $P(X \geq 5)$;
- $P(X \geq 10)$;

- $P(X \geq 20)$;
- $P(X \geq 10^{10})$.

Now, find an upper bound for

- $P(|X-10| \geq 10) = P(\{X \geq 20\} \cup \{X \leq 0\})$.

What do you notice?
Now I also tell you that this random variable has variance $\sigma^2 := V(X) = 1$. Find an upper bound for

- $P(|X-10| \geq 2)$.

How does it compare to Markov's?
Finally, I tell you that the pdf of X is

$$f(x) = (x-9)\mathbb{1}_{[9,10]}(x) + (11-x)\mathbb{1}_{(10,11]}(x).$$

- *Compute* $P(|X-10| \geq 2)$.

Solution:
First,

- $P(X \geq 5) \leq 2$. This is really a non-informative bound since we know that it can not be larger than 1 either.
- $P(X \geq 10) \leq 1$. Not much information in this bound either.
- $P(X \geq 20) \leq 1/2$. Finally something less than 1!
- $P(X \geq 10^{10}) \leq 1/(10^9) \approx 0$, since the mean is finite I know that X cannot take extremely high values.

As you see, Markov's inequality is not informative unless we ask the probability of an event $\{X \geq b\}$ where $b \geq E[X]$. Why?
Second,

- $P(|X-10| \geq 10) \leq P(X \geq 20) + P(X \leq 0) = P(X \geq 20) \leq 1/2$.

Given the variance, let's use Chebychev.[1]

- $P(|X-10| \geq 10) \leq 1/10^2 = 0.01$.

Finally,

- This is clearly zero!

[1] https://www.quora.com/What-is-the-preferred-spelling-of-Chebyshev-Chebyshev-Chebishev-Tchebyscheff-