Homework 6 (due Friday, June 14th at 11:59pm)

1. (10 pts) Let X, Y, Z be three random variables such that $E(X|Z) = 3Z$, $E(Y|Z) = 1$, and $E(XY|Z) = 4Z$. Assume furthermore that $E(Z) = 0$. Are X and Y uncorrelated?

   Solution:
   We have

$$\begin{aligned} Cov(X,Y) &= E[Cov(X,Y|Z)] + Cov[E(X|Z), E(Y|Z)] \\ &= E[E(XY|Z) - E(X|Z)E(Y|Z)] + Cov(3Z, 1) \\ &= E(4Z - 3Z * 1) + 3Cov(Z, 1) = E(Z) + 3 * 0 = 0 + 0 = 0. \end{aligned}$$

   It follows that X and Y are uncorrelated.

2. (15 pts) Let the r.v. $X \sim$Poisson(λ) with $\lambda \in \mathbb{N}_+$. Let $Y = (X - \lambda)^2$. Find the pmf of Y.
   Solution:
   It is clear that $p_Y(y) = 0$ for $y < 0$. Then, for $\sqrt{y} \in \mathbb{N}_+$,

$$\begin{aligned} p_Y(y) &= P(Y = y) = P((X - \lambda)^2 = y) \\ &= P(X - \lambda = -\sqrt{y}) + P(X - \lambda = \sqrt{y}) \\ &= P(X = \lambda - \sqrt{y}) + P(X = \lambda + \sqrt{y}) \end{aligned}$$

   Therefore

$$p_Y(y) = \begin{cases} \frac{\lambda^{\lambda+\sqrt{y}} e^{-\lambda}}{(\lambda+\sqrt{y})!} \text{ if } \sqrt{y} \in \mathbb{N}_+ \text{ and } \lambda < \sqrt{y} \\ \frac{\lambda^{\lambda+\sqrt{y}} e^{-\lambda}}{(\lambda+\sqrt{y})!} + \frac{\lambda^{\lambda-\sqrt{y}} e^{-\lambda}}{(\lambda-\sqrt{y})!} \text{ if } \sqrt{y} \in \mathbb{N}_+ \text{ and } \lambda \geq \sqrt{y} \\ 0 \text{ o/w} \end{cases}$$

   Here is some code to perform this experiment via simulation.

```
require(ggplot2)
n <- 1e5
lambda <- 10
X <- rpois(n, 10)
dat <- data.frame(sample = c((X-lambda)^2, X),
        Group = c(rep('dist y',n),rep('dist x',n)))

ggplot(dat, aes(x=sample, fill=Group)) +
  geom_histogram(aes(y=..count../sum(..count..)),
        binwidth=1, alpha=.8, position="identity") +
  xlim(0,100)
```

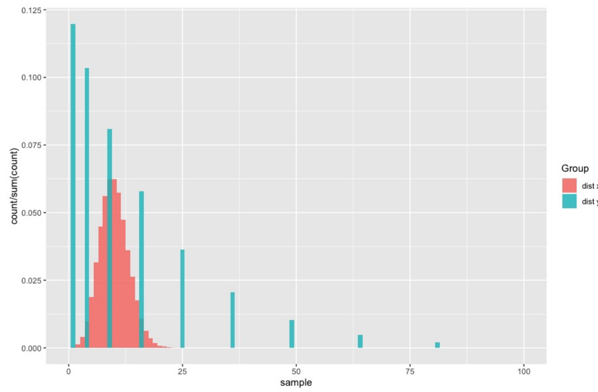

Figure 1: Distributions for exercise 1.

3. (10 pts) Let the r.v. X have pdf

$$f_X(x) = 4x^3 \mathbb{1}(0 < x \leq 1).$$

Let $Y = (X+1)/X$. Find the pdf of Y.

Solution:
First, notice that $Y \in [2, \infty)$. Therefore for $y \geq 2$,

$$P(Y \leq y) = P(X + 1 \leq yX)$$
$$= 1 - P\left(X \leq \frac{1}{y-1}\right) = 1 - F_X\left(\frac{1}{y-1}\right) = 1 - \left(\frac{1}{y-1}\right)^4.$$

Therefore the pdf is

$$f_Y(y) = \begin{cases} 4\left(\frac{1}{y-1}\right)^5 \text{ for } y \geq 2 \\ 0 \text{ o/w} \end{cases}$$

4. (10 pts) Let $X \sim f_X$ with

$$f_X(x) = \frac{b}{x^2} \mathbb{1}_{[b,\infty)}(x)$$

and let $U \sim \text{Uniform}(0,1)$. Find a function g such that $g(U)$ has the same distribution of X.

Solution:
Using the result of exercise 5, we need to set $g = F_X^{-1}$. We have

$$F_X(x) = \begin{cases} 0 \text{ if } x < b \\ \int_b^x f_X(y)\, dy \text{ if } x \geq b \end{cases} = \begin{cases} 0 \text{ if } x < b \\ \int_b^x \frac{b}{y^2}\, dy \text{ if } x \geq b \end{cases}$$
$$= \begin{cases} 0 \text{ if } x < b \\ -\left.\frac{b}{y}\right|_b^x \text{ if } x \geq b \end{cases} = \begin{cases} 0 \text{ if } x < b \\ 1 - \frac{b}{x} \text{ if } x \geq b. \end{cases}$$

It follows that

$$g(x) = F_Y^{-1}(x) = \frac{b}{1-x}.$$

5. (20 pts) Let $X_1, X_2 \overset{iid}{\sim}$ Uniform(0,1). Compute $E[X_1/X_2]$.

Solution:
<u>(Solution 1)</u>:

$$E\left[\frac{X_1}{X_2}\right] = E\left[X_1\right] E\left[\frac{1}{X_2}\right]$$

where

$$E\left[\frac{1}{X_2}\right] = \int_0^1 \frac{1}{x} dx = +\infty$$

hence

$$E\left[\frac{1}{X_2}\right] = +\infty.$$

<u>(Solution 2)</u>:

Alternatively, we can go with the hard way.[1] For every $x \in (0, \infty)$:

$$P\left(\frac{X_1}{X_2} \le x\right) = \mathbb{E}_{X_1}[\mathbb{E}_{X_2}[\mathbb{1}(X_1 \le xX_2)|X_2]]$$
$$= \int_0^1 \int_0^1 \mathbb{1}(x_1 \le xx_2) dx_1 dx_2 = \int_0^1 \int_0^{\min\{xx_2, 1\}} dx_1 dx_2$$
$$= \int_0^1 xx_2 \mathbb{1}(xx_2 \le 1) dx_2 + \int_0^1 \mathbb{1}(xx_2 > 1) dx_2$$
$$= \int_0^1 xx_2 \mathbb{1}(x_2 \le 1/x) dx_2 + \int_0^1 \mathbb{1}(x_2 > 1/x) dx_2$$
$$= \int_0^{\min\{1, 1/x\}} xx_2 dx_2 + \mathbb{1}(x > 1) \int_{1/x}^1 dx_2$$
$$= \mathbb{1}(x > 1) \left(x \int_0^{1/x} x_2 dx_2 + \int_{1/x}^1 dx_2\right) + \mathbb{1}(x \le 1) x \int_0^1 x_2 dx_2$$
$$= \left(1 - \frac{1}{2x}\right) \mathbb{1}(x > 1) + \frac{x}{2} \mathbb{1}(x \le 1).$$

Therefore the pdf is

$$f_{X_1/X_2}(x) = \frac{1}{2x^2} \mathbb{1}(x > 1) + \frac{1}{2} \mathbb{1}(0 < x \le 1).$$

Therefore, calling $Y = X_1/X_2$,

$$E[Y] = \int_1^\infty \frac{1}{2y} dy + \frac{1}{2} = \frac{1}{2} \log(y)|_1^\infty + \frac{1}{2} = \infty.$$

6. (20 pts) Consider the scores (random variables) of the student in this class $X_1, \ldots, X_n$. For now consider them to be continuous. We have good reasons to believe that the following versions of the scores

$$Y_i = \frac{X_i}{100}$$

for $i = 1, \ldots, n$ be distributed, identically and independently, as Beta with parameters $\alpha > 0$ and $\beta = 1$, that is $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Beta}(\alpha, 1)$.

   (a) Before receiving the actual result, you would like to know what's the probability that you will get a score higher than 80, that is $P(X > 80)$. Compute it, knowing that $\alpha = 2$.

[1] However, there is no need since the r.v.'s are independent.

(b) Given that $x_1 = \cdots = x_{n-1} = 60$, what is $P(X_n > 80)$? Again, you know that $\alpha = 2$.

(c) Now let $\alpha \in \mathbb{R}_+$, while β is still 1. You have received your scores, $x_1, \ldots, x_n$ and you want to find out

- what kind of shape this Beta distribution has, that is find the MLE of α given $x_1, \ldots, x_n$.
- what is the mean θ of the scores, given that the distribution of Y is a Beta$(\hat{\alpha}, \beta)$ with $\hat{\alpha}$ being the MLE. Compute it for $n = 4$ and $\mathbf{x} = (90, 90, 90, 80)$.
- If Riccardo decides to change the mean θ (of the distribution) to $\theta' = \theta + (1-\theta)/2$ to increase the grades, what is the MLE of θ'?

Hint (1) 1: you probably need to consider the log-likelihood for $y_1, \ldots, y_n$ (the transformed scores).

$$\ell(y_1, \ldots, y_n | \alpha, \beta) = \log \prod_{i=1}^{n} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}.$$

where $y_1, \ldots, y_n$ are $y_i = x_i/100$ for $i = 1, \ldots, n$.
Hint (2/3): here you need to apply the same (invariant) property.

Solution:

(a)

$$P(X > 80) = P(Y > 0.8) = 1 - \int_0^{4/5} x dx = 1 - \frac{8}{25} = \frac{17}{25}.$$

(b) It's still 17/25!!! The draws are independent and identically distributed.

(c)
- Consider $y_1, \ldots, y_n$ where $y_i = x_i/100$ for $i = 1, \ldots, n$. Now the pdf of Y is

$$f_Y(y; \alpha) = \alpha y^{\alpha-1} \mathbb{1}(y \in [0,1])$$

Therefore for $y \in [0,1]$,

$$\frac{\partial}{\partial \alpha} \sum_{i=1}^{n} \log(\alpha y_i^{\alpha-1}) = \frac{\partial}{\partial \alpha} \left[n \log \alpha + \sum_{i=1}^{n} (\alpha - 1) \log y_i \right]$$

$$= \frac{n}{\alpha} + \sum_{i=1}^{n} \log y_i \implies \hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \log y_i}.$$

- We can compute $\hat{\alpha}$ for this data, and we obtain $\hat{\alpha} \approx 7.4$. Now, by the invariance property of the MLE, we have

$$\theta = \frac{\hat{\alpha}}{\hat{\alpha}+1} \cdot 100 \approx 88.$$

  Compare it to the mean (without knowledge of the Beta distribution), that is 87.5!
- Again, here we use the invariance property of the MLE. The result is $\hat{\alpha}' \approx 0.94$.

7. (15 pts) Let $x_1, \dots, x_n$ be the realizations of $X_1, \dots, X_n \overset{iid}{\sim} f_X$ with

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}(x \in [0, \infty)).$$

- Find the MLE of β. [2]
- Find the MLE of $\beta' = \beta^2$.
- Find the MLE of λ for $x_1, \dots, x_n$ when $X_1, \dots, X_n \overset{iid}{\sim} f_X$ with

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}(x \in [0, \infty)).$$

  You can use the results from the previous steps.[3]

Solution:

- Let's find the MLE for β first.

$$\frac{\partial}{\partial \beta}\left(n\alpha \log \beta - n \log \Gamma(\alpha) + (\alpha - 1)\sum_{i=1}^n \log x_i - \beta \sum_{i=1}^n x_i \right) = 0$$

$$\implies \frac{n\alpha}{\beta} - \sum_{i=1}^n x_i = 0 \implies \hat{\beta} = \frac{n\alpha}{\sum_{i=1}^n x_i}.$$

- Using the invariance property, we obtain

$$\hat{\beta}' = \left(\frac{n\alpha}{\sum_{i=1}^n x_i} \right)^2.$$

- Since $\mathrm{Exp}(\beta) = \mathrm{Beta}(1, \beta)$, we have

$$\hat{\beta} = \frac{n}{\sum_{i=1}^n} x_i.$$

[2]Notice that the MLE for α is much harder to find due to the term $\log \Gamma(\alpha)$.
[3]Given that we have already done it in class, this should be straightforward.