Homework 7 (due Tuesday, June 18th at 11:59pm)

This homework replicates the content of the exam.

1. (15 pts) Let the joint pdf of the r.v.'s (X, Y) be

$$f_{X,Y}(x, y) = \begin{cases} cyx^2 \text{ if } 0 < x < 5,\ 0 < y < x \\ 0 \text{ o/w} \end{cases}$$

   - (5 pts) Compute the normalizing constant c.
   - (5 pts) Compute f_X.
   - (5 pts) Compute $P(X \in [2, 5])$.
   - (5 pts) Compute $f_{Y|X=x}$ for $0 < x < 5$.

2. (20 pts) Let $X_1 \sim \text{Uniform}(0, 1)$ and $X_2 \sim \text{Uniform}(0, 2)$.[1] Moreover, $X_1 \perp\!\!\!\perp X_2$. Compute $F_Y(y)$ where $Y = X_1 - X_2$ and $y \in \mathbb{R}$.

3. (15 pts) Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. Consider the realizations of those r.v.'s, that is $X_1 = x_1, \ldots, X_n = x_n$.

   - (5 pts) Find the MLE for μ and the MLE for σ^2.
   - (10 pts) Prove that the MLE estimator for σ^2 is biased.[2]
   - (5 pts) Finally, find the MLE for $\sigma = \sqrt{\sigma^2}$.[3]

4. (15 pts) Let $X_1 \sim \text{Beta}(\alpha_1, \beta_1)$, $X_2 \sim \text{Bernoulli}(X_3)$ where $X_3 \sim \text{Beta}(\alpha_3, \beta_3)$. You also know that $X_1 \perp\!\!\!\perp (X_2, X_3)$.

   - (5 pts) Compute $E[X_1]$.[4]
   - (5 pts) Compute $E[X_2|X_3]$.
   - (5 pts) Compute $E[X_1 X_2]$.[5]

---

[1] Pay attention to the different supports!

[2] In order to prove this fact, you need to replace μ in the formula of $\hat{\sigma}^2$ with $\hat{\mu}$, its MLE. Moreover, it would be useful to remember that $V(Y) = E[Y^2] - (E[Y])^2$.

[3] Remember the properties of the MLE.

[4] Since you can find the solution in the notes, I already help you with it: $E[X_1] = \frac{\alpha_1}{\alpha_1 + \beta_1}$. However, you still need to prove it!

[5] You might need the following implication (proved in hw 5):

$$X_1 \perp\!\!\!\perp (X_2, X_3) \implies \begin{cases} X_1 \perp\!\!\!\perp X_2 | X_3 \\ X_1 \perp\!\!\!\perp X_3 | X_2 \end{cases}$$

- (5 pts) Compute $V(X_1 + X_2)$.[6]

5. (15 pts) Let $Y = \exp\{\mu + \sigma Z\}$ where $Z \sim \mathcal{N}(0,1)$ and $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$.

   - (10 pts) Show that[7]

   $$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}}.$$

   - (5 pts) Compute $E[Y]$.[8]
   - (5 pts) Compute $V(Y)$.

6. (20 pts) Let the r.v. X have pdf

   $$f_X(x) = \frac{1}{10}\mathbb{1}(x \in [0,10]).$$

   Now let $Y|X \sim \text{Bernoulli}(m^*(X))$ where $m^*(x) = x/(x+1)$. Moreover, let $h(X) = \mathbb{1}(m(X) > 1/2)$ where $m(X) = x/(x+2)$.

   - (5 pts) Compute $E[Y]$ and $E[h(X)]$.[9]
   - (5 pts) Compute $E[h(X)|Y = 0]$. [10]
   - (5 pts) Compute $E[(h(X) - Y)^2]$.[11]
   - (5 pts) Compute $P(X \in [2,6]|h(X) = 1)$.

---

[6] These computations might get quite messy (but not difficult), so leave them at the end. As long as you break down the variance correctly, and are able to rewrite everything in terms of the integral, you will get full score. Again, you can find the result in the notes so I'll help you with that: $V(X_1) = \frac{\alpha_1\beta_1}{(\alpha_1+\beta_1)^2(\alpha_1+\beta_1+1)}$.

[7] You *might* need the following fact from the previous homework:

$$\frac{\partial\phi(g(y))}{\partial y} = f_Z(g(y))\frac{\partial g(y)}{\partial y}.$$

[8] You *will need* the following fact, shown in the homework,$E[e^{tZ}] = e^{\frac{t^2}{2}}$. Do not try to integrate Y. Instead, use linearity of the integral togerther with this fact.

[9] You will need the following fact: $\int \frac{x}{c+x}dx = x - c\log(x+c)$ where c is some constant larger than 0, and $x > 0$.

[10] Remember that $E[h(X)] = P(h(X) = 1)$ for $h \in \{0,1\}$ and Bayes theorem might be useful here.

[11] You might want to use linearity of the integral after opening the square.