Homework 9 (due Monday, June 24th at 11:59pm)

1. (5 pts) I forgot to add this problem in one of the early problem sets, so here we go. It's one of the most famous paradoxes in Statistics. You are at the casino, and you play the following game: a coin is tossed (H vs T) repeatedly until head appears (the game stops), and at every toss the amount you win is doubled. That is, for instance,

   - sequence is (H) - you win nothing
   - sequence is (T,T,H) - you win 2+2· 2=2+4=6
   - sequence is (T,T,T,T,T,T,T,T,H) -
you win 2+4+8+16+64+128+236+472=510

   What's the maximum price that you would pay to enter this game?
   Hint: The expected value of the winning might be useful.

   Solution:
   This is called the St. Petersburg paradox. Let X be the r.v. representing the winning. Then

   $$E[X] = 2\frac{1}{2} + 4\frac{2^2}{+}\frac{8}{2^3} + \cdots = 1 + 1 + 1 + \cdots = \infty.$$

   Therefore you *should* be willing to pay any finite price in order to play since you would always gain in expectation. However, you probably would have payed no more than 20 dollars for this game… until you computed the expected value.

2. (5 pts) We have seen in class that for $n \in \mathbb{Z}^+$, we have

   $$n! = \int_0^\infty e^{-x} x^n dx.$$

   Prove it.
   Hint: it's just integration by parts.

   Solution:
   Integrating by parts we have

   $$\begin{aligned} \int_0^\infty e^{-x} x^n dx &= -e^{-x} x^n |_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx \\ &= \cdots = n(n-1)\ldots 2 \cdot 1 \int_0^\infty e^{-x} dx = n! \end{aligned}$$

3. (5 pts) In class we have seen Taylor expansions. Let's do some exercises on this topic to get a better grasp of it.

   - Prove that
   $$\frac{1}{e} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$$
   - Show the following Maclaurin series[1]
   $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots$$
   - Show that the Taylor series of the natural logarithm around 1 is
   $$\log(x) = -1 + x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$
   Now show also that the Maclaurin series of the natural logarithm is
   $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Solution:

   - If we take a Taylor expansion of e^{-x} around 0, and then choosing $x = -1$,
   $$\frac{1}{e} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = -1 + \frac{1}{2!} - \frac{1}{3!} + \dots.$$
   - Consider the Taylor expansion of $f(x) = (1+x)^{-1}$ around 0:
   $$(1+x)^{-1} = 1 + (-1)x + \frac{2}{2!}x^2 + \frac{-3 \cdot 2}{3!}x^3 + \dots = \sum_{i=0}^{\infty}(-1)^i x^i.$$
   - Consider the Taylor expansion of $\log(x)$ around a,
   $$\log x = \log a + \frac{1}{a}(x-a) - \frac{1}{a^2}\frac{(x-a)^2}{2!} + \frac{2}{a^3}\frac{(x-a)^3}{3!} + \dots$$
   and, taking $a = 1$ we obtain
   $$\log x = -1 + x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

[1]Notice that this name should ring a bell: we are taking a Taylor expansion around $a = 0$!

Now, for the second part we need to take a Taylor expansion around 0, that yields

$$\log x = -1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

4. (10 pts) Prove the following statements.

   - Using Taylor series, prove that for a continuous function f and some random variable X, we have

     $$V(f(X)) \approx (f'(E[X]))^2 V(X)$$

     using a first-order approximation.[2]

   - Let $\hat{p}$ be the MLE estimator for p, the parameter of $X \sim \text{Binomial}(n,p)$, that is $\hat{p} = X/n$. Apply the result above using $f(p) = p/(1-p)$ (the odds) and $X = \hat{p}$. Compute it for $p = 0.3$ and $n = 3$.

   - Moreover, using again a first-order approximation, prove that

     $$E[f(X)] \approx f(E[X])$$

     and compute it plugging in the values indicated above.

   The last two results should remind you of the Delta method!

   Solution:

   - Consider a first-order Taylor expansion of f around $E[X]$,

     $$V(f(X)) \approx V(f(E[X]) + f'(E[X])X)$$
     $$= V(f(E[X])) + V(f'(E[X])X) = (f'(E[X]))^2 V(X).$$

   - Now, we see that

     $$f'(p) = \frac{1}{(1-p)^2}$$

     and therefore

     $$V\left(\frac{\hat{p}}{1-\hat{p}}\right) \approx \frac{1}{(1-p)^4} V(\hat{p}) = \frac{1}{(1-p)^4}\frac{p(1-p)}{n} = \frac{p}{(1-p)^3 n} \approx 0.29.$$

[2] That is, we do not consider second-order terms.

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$$E[f(X)] \approx E[f(E[X]) + f'(E[X])(X - E[X])]$$
$$= f(E[X]) + f'(E[X])(E[X] - E[X]) = f(E[X]).$$

Its value is 3/7.

5. (10 pts) Using the central limit theorem for Poisson random variables, compute the value of
$$\lim_{n\to\infty} e^{-n} \sum_{i=0}^{n} \frac{n^i}{i!}.$$

Solution:
If you denote $X_1, \cdots, X_n$ each i.i.d. Poisson random variables with parameter 1. Then, S_n is Poisson(n), and the expectation of S_n is $E(S_n) = n$. The central limit theorem states that the sum of Poissons should go to $P(S_n \le n) \to \frac{1}{2}$. Lastly, notice that the quantity of interest is exactly $P(S_n \le n)$!

6. (10 pts) Let $\{X_t\}_{t\ge 0}$ be a Poisson process with parameter λ. For $0 < s < t$, compute
$$P(X_s = 1|X_t = 1).$$

Solution:

$$P(X_s = 1|X_t = 1) = \frac{P(X_s = X_t = 1)}{P(X_t = 1)} = \frac{s}{t}.$$

7. (15 pts) Let $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ), and $X \perp\!\!\!\perp Y$. Consider $Z = X + Y$.

   - Although we have already shown this in class, prove again that $X + Y \sim$ Poisson($\lambda + \mu$) using mgf's.
   - Compute the probability that $X = 1|Z = 1$.
   - What's the distribution of $X|Z = z$ for $z > 0$?

Solution:

- This result has already been derived in class.

- $$\begin{aligned} P(X=1|Z=1) &= P(X=1,Y=0|Z=1) \\ &= \frac{P(X=1,Y=0)}{P(Z=1)} = \frac{P(X=1)P(Y=0)}{P(Z=1)} \\ &= \frac{\lambda}{\lambda+\mu}. \end{aligned}$$

- From the previous fact we can simply conclude that

$$X \sim \text{Binomial}(z, \lambda/(\lambda+\mu)).$$

8. (15 pts) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Binomial}(X, p)$. Compute $E[Y]$. To make it more real life, think about X being the number of people in line, and Y being the number of people in line that are also tired. Compute the expected number of people that are tired and in line for $p = 0.8$ and $\lambda = 10$.
Hint: rewrite $Y = \sum_{i=1}^{X} Z_i = \sum_{i=1}^{\infty} Z_i \mathbb{1}(i \leq X)$ where $Z_i \sim \text{Bernoulli(p)}$. Moreover, remember that $E[X] = \sum_{i=1}^{\infty} P(X \geq i)$.

Solution:

$$\begin{aligned} E[Y] &= E[\sum_{i=1}^{\infty} Z_i \mathbb{1}(i \leq X)] = \sum_{i=1}^{\infty} E[Z_i \mathbb{1}(i \leq X)] \\ &= \sum_{i=1}^{\infty} E[Z_i]E[\mathbb{1}(i \leq X)] = p\sum_{i=1}^{\infty} P(X \geq i) = p\lambda. \end{aligned}$$

Therefore we have $E[Y] = 8$.

9. (10 pts) Let $\{X_t\}_{t\geq 0}$ be a Poisson process with parameter λ.

- Let $T_1, T_2, \ldots$ be the arrivals in the time window $[0, T]$. Prove that $E[X_T T - \sum_{i=1}^{X_T} T_i] = \frac{T^2}{2}\lambda$. In a real life example, a train leaves at time T, and this is the expected total time that all people, arriving to the station according to a Poisson distribution, will have to wait for. Compute it for $T = 10$ minutes and $\lambda = 15$.
Hint: use the law of total probability conditioning on X_T. Moreover, remember that

$$E[\sum_{i=1}^{X_T} T_{(i)} | X_T = n] = E[\sum_{i=1}^{X_T} T_i | X_T = n] = \sum_{i=1}^{n} E[T_i] = nT/2$$

since $\sum_{i=1}^{X_T} T_{(i)} = \sum_{i=1}^{X_T} T_i$, that is the sum of the first X_T order statistics is equal to the sum of the first X_T random variables.

- Now, let $S < T$. Prove that

$$E\left[(X_T - X_S)T + X_S S - \sum_{i=1}^{X_T} T_i\right] = \frac{\lambda S^2}{2} + \frac{\lambda(T-S)^2}{2}.$$

  Again, this is asking the expected amount of time waited if two different trains left at times S and T, with $S < T$.

- Now let $T \sim \text{Uniform}(0, M)$. Compute again

$$E[X_T T - \sum_{i=1}^{X_T} T_i].$$

  Hint: integrate over T.

Solution:

- $$\begin{aligned} E[T - \sum_{i=1}^{X_T} T_i] &= \sum_{i=0}^{\infty} E[X_T T - \sum_{i=1}^{X_T} T_i | X_T = i] P(X_T = i) \\ &= \sum_{i=0}^{\infty} i \frac{T}{2} P(X_T = i) = \sum_{i=1}^{\infty} i \frac{T}{2} P(X_T = i) \\ &= \frac{T}{2} E[X_T] = \frac{T^2}{2}\lambda \end{aligned}$$

  For these values we obtain $50 \cdot 15 = 750$ minutes, such a waste of time!

- $$\begin{aligned} &E\left[(X_T - X_S)T + X_S S - \sum_{i=1}^{X_T} T_i\right] \\ &= E\left[X_{T-S}(T-S) - \sum_{i=1}^{X_{T-S}} T_i\right] + E\left[X_S S - \sum_{i=1}^{X_S} T_i\right] \\ &= \frac{\lambda(T-S)^2}{2} + \frac{\lambda S^2}{2}. \end{aligned}$$

  thanks to the independent and stationary increments.

- $$E[X_T T - \sum_{i=1}^{X_T} T_i] = \int_0^M \frac{1}{M} E[X_T T - \sum_{i=1}^{X_T} T_i | T = t] dt$$
$$= \frac{\lambda}{2M} \int_0^M t^2 dt = \frac{\lambda}{2M} \frac{M^3}{3} = \frac{\lambda M^2}{6}.$$

10. (15 pts) Let $T_1 \sim \text{Exponential}(\lambda_1)$ and $T_2 \sim \text{Exponential}(\lambda_2)$. Compute $P(T_1 < T_2 + T)$ where $T > 0$.
As a real life example, think about your laptop (hopefully T is very large) vs your friend's laptop's lifetime and compute the probability that yours breaks at most one year after his. Consider $T = 1, \lambda_1 = 2, \lambda_2 = 1$.
Hint: First, use the law of total probability with the event $\{T_1 < T\} \cup \{T_1 \geq T\}$. In order to get the result, you will need to compute

$$P(T_1 < T_2) = \int_0^\infty P(T_1 < T_2 | T_2 = t_2) f_{T_2}(t_2) dt_2$$

where f_{T_2} is the pdf of T_2.

Solution:

$$\begin{aligned} P(T_1 < T_2 + T) &= P(T_1 < T) + P(T_1 \geq T) P(T_1 < T_2 + T | T_1 \geq T) \\ &= 1 - e^{-\lambda_1 T} + e^{-\lambda_1 T} P(T_1 - T < T_2 | T_1 \geq T) \\ &= 1 - e^{-\lambda_1 T} + e^{-\lambda_1 T} P(T_1 < T_2) \end{aligned}$$

Now, let's compute

$$\begin{aligned} P(T_1 < T_2) &= \int_0^\infty P(T_1 < T_2 | T_2 = t_2) f_{T_2}(t_2) dt_2 \\ &= \int_0^\infty (1 - e^{-\lambda_1 t_2}) \lambda_2 e^{-\lambda_2 t_2} dt_2 \\ &= 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1 + \lambda_2) t_2} dt_2 \\ &= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

hence

$$P(T_1 < T_2 + T) = 1 + e^{-\lambda_1 T} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} - 1 \right).$$

For the values indicated, we obtain $P(T_1 < T_2 + 1) \approx 95\%$.