# Quiz 3 (Friday June 7th)

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Total: 120 points. Full score: 100 points.

1. (20 pts) Let $Y \sim \text{Beta}(\alpha, \beta)$ for $\alpha, \beta > 0$, and $X = Y + 10$. Compute $P(X \leq 11)$.

2. (20 pts) Let the r.v. $X$ have pmf conditional on the .r.v $Y = y > 0$,

$$p_{X|Y=y}(x) = \frac{y^x e^{-y}}{x!} \text{ for } x = 0, 1 \ldots$$

Compute $E[X|Y = y]$.

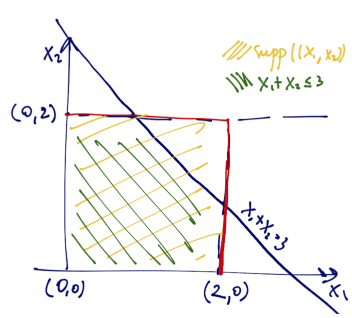
3. (20 pts) Let $X_1, X_2$ be two independent r.v.'s with pdf

$$f(x) = \frac{1}{2}\mathbb{1}_{[0,2]}(x).$$

Compute $P(X_1 + X_2 \leq 3)$.

**Solution:** thanks to one of you, I found out that my "no-math" solution was actually wrong! For this reason, I include here the way you could have solved this problem.

Solution 1:
If you draw the support the solution is straightforward. Indeed, the function is constant on the support. It follows that the area of the



support is $2 \cdot 2 = 4$, and the region that we are interested in has area $7/2$. Consequently

$$P(X_1 + X_2 \leq 3) = 1 - P(X_1 + X_2 > 3) = 1 - \frac{4 - 3.5}{4} = \frac{7}{8}.$$

<u>Solution 2</u>:
Let $Y = X_1 + X_2$. Let's make this problem more general. [1]We need to compute $F_Y(y)$. For $y \leq 0$ we already know that $F_Y(y) = 0$, and for $y > 4$ we know that $F_y(y) = 1$. Therefore, for some $y \in [0, 4]$,

$$\begin{aligned}
F_Y(y) &= E_Y[\mathbb{1}(Y \leq y)] \\
&= E_{X_1,X_2}[\mathbb{1}(X_1 + X_2 \leq y)] \\
&= E_{X_1}[E_{X_2}[\mathbb{1}(X_1 + X_2 \leq y)|X_1]] \\
&= \int_0^2 E_{X_2}[\mathbb{1}(X_1 + X_2 \leq y)|X_1 = x_1]\frac{1}{2}dx_1 \\
&= \int_0^2 \int_0^2 \mathbb{1}(x_1 + x_2 \leq y)\frac{1}{4}dx_2 dx_1
\end{aligned}$$

Now you need to be careful with the extrema of the integral.

$$\begin{aligned}
&= \frac{1}{4}\int_0^2 \int_0^2 \mathbb{1}(x_2 \leq y - x_1)dx_2 dx_1 \\
&= \frac{1}{4}\int_0^2 \int_0^2 \mathbb{1}(x_2 \leq y - x_1)[\mathbb{1}(y < 2) + \mathbb{1}(y \geq 2)]dx_2 dx_1
\end{aligned}$$

Now, separate the integrals by linearity. Let's start with the second integrals.

$$\begin{aligned}
&\frac{1}{4}\mathbb{1}(y \geq 2)\int_0^2 \int_0^{\min\{y - x_1, 2\}} dx_2 dx_1 \\
&= \frac{1}{4}\mathbb{1}(y \geq 2)\int_0^2 [(y - x_1)\mathbb{1}(x_1 \geq y - 2) + 2\mathbb{1}(x_1 < y - 2)]dx_1 \\
&= \frac{1}{4}\mathbb{1}(y \geq 2)\left[\int_{y-2}^2 (y - x_1)dx_1 + 2\int_0^2 dx_1\right] \\
&= \frac{1}{4}\mathbb{1}(y \geq 2)\left[-\frac{y^2}{2} + 4y - 4\right]
\end{aligned}$$

[1]For the quiz this was not required.

For instance, now we have

$$F_Y(3) = \frac{1}{4}\left(-\frac{9}{2} + 12 - 4\right) = \frac{7}{8}, \text{ and } F_Y(4) = 1.$$

For the first integral, we have

$$\begin{aligned} &\frac{1}{4}\mathbb{1}(y < 2)\int_0^y \int_0^{y-x_1} dx_2 dx_1 \\ &= \frac{1}{4}\mathbb{1}(y < 2)\int_0^y (y - x_1)dx_1 \\ &= \frac{1}{4}\mathbb{1}(y < 2)\frac{y^2}{2}. \end{aligned}$$

For instance, $P(X_1 + X_2 \leq 1) = 1/8$. Clearly, $P(X_1 + X_2 \leq 0) = 0$. Finally, we can rewrite the cdf as

$$F_Y(y) = \begin{cases} 0 \text{ if } y < 0 \\ \frac{y^2}{8} \text{ if } 0 \leq y \leq 2 \\ \frac{1}{4}\left[-\frac{y^2}{2} + 4y - 4\right] \text{ if } 2 < y \leq 4 \\ 1 \text{ o/w} \end{cases}$$

4. (20 pts) Let $X_1 \sim \text{Bernoulli}(p)$. Compute $E[X_2]$ where $X_2 \sim \text{Bernoulli}(X_1)$.

5. (20 pts) Let $X, Y$ be two r.v.'s such that $X \perp\!\!\!\perp Y$. Moreover, $X \sim \mathcal{N}(0, 1)$ and $Y \sim \text{Poisson}(\lambda)$. Compute $V(X - Y)$.

6. (20 pts) Let the joint distribution of $(X, Y)$ be

$$f_{X,Y}(x, y) = c\mathbb{1}(2 \leq x \leq 10, 3 \leq x \leq 6y)$$

for some normalizing constant $c$. Compute $f_X(x)$.