Test I (Friday May 31st, time allowed: 80 minutes)

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Total: 120 points. Full score: 100 points.

1. (20 pts) Let $X \sim \mathcal{N}(10, 1)$, that is X is distributed as a normal with mean 10 and variance 1.

   - Compute $F_X(10)$.
   - Compute $P(9 \leq X \leq 11)$. You might need the fact that $\phi(1) = 0.8413$ (ϕ is the cumulative density function of the standard normal).
   - Let $Y \sim \mathcal{N}(20, 1)$. Compute $P(20 \leq Y \leq 21)$.
   - Let $Z = \sum_{i=1}^{n} X_i^2$ where $X_i \sim \mathcal{N}(0, 1)$ for $i = 1, \ldots, n$ and $X_i \perp\!\!\!\perp X_j$ for $j \neq i$. Compute $E[Z]$.

   Solution:

   - (5 pts) Write $X_0 = X - 10$. Then we know $X_0 \sim \mathcal{N}(0, 1)$.

     $$F_X(10) = \mathbb{P}(X \leq 10) = \mathbb{P}(X_0 \leq 0) = F_{X_0}(0) = \phi(0),$$

     where $\phi(\cdot)$ is c.d.f. for $\mathcal{N}(0, 1)$. Recall the fact $\phi(-x) = 1 - \phi(x)$, we know $\phi(0) = 1/2$. Then

     $$F_X(10) = \frac{1}{2}.$$

   - (5 pts) Using same definition of X_0, we have

     $$\mathbb{P}(9 \leq X \leq 11) = \mathbb{P}(-1 \leq X_0 \leq 1) = \phi(1) - \phi(-1) = 2\phi(1) - 1 = 0.6826.$$

   - (5 pts) Similarly, we have $Y - 20 \sim \mathcal{N}(0, 1)$. Then

     $$\mathbb{P}(20 \leq Y \leq 21) = \mathbb{P}(0 \leq Y - 20 \leq 1) = \phi(1) - \phi(0) = 0.3413.$$

   - (5 pts) There are two ways to solve the problem.
     You should notice that $Z \sim \chi^2(n)$. Then, from the notes, $E[Z] = n$.

Alternatively, recall that $V(X) = E[X^2] - E[X]^2$, hence $E[X^2] = V(V) + E[X]^2$. Therefore

$$E[Z] = \sum_{i=1}^{n} E[X_i^2] = \sum_{i=1}^{n}(V(X_i) + E[X_i]^2) = \sum_{i=1}^{n}(1+0) = n.$$

Remark: partial credit is given to reasonable attempts. Make sure that you correctly understand the last part.

2. (20 pts) Yue and Riccardo are part of a group of n people who are about to receive some seating instructions.

   - If the n people are randomly seated in a line, what is the probability that Yue and Riccardo are next to each other?
   - What would the probability be if the people were randomly arranged in a circle?

   Solution:

   - (10 pts)
$$\frac{2(n-1)(n-2)!}{n!} = \frac{2}{n}$$
   - (10 pts)
$$\frac{2n(n-2)}{n!}$$

   Remark: We give partial credit depending on how close you are to correct answer.

3. (20 pts) Prove that

- if $P(A|B) = P(A|B^c)$, then A and B are independent.
  Hint: you *might* find useful to remember the fact that
  $$\frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2} \text{ if } b_1 + b_2 \neq 0.$$
- if $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$, then $P(A) > P(B)$.

Solution:

- (10 pts) Solution 1: (Riccardo)
  $$\begin{aligned} P(A)P(B) &= [P(A \cap B^c) + P(A \cap B)]P(B) \\ &= [P(A|B^c)P(B^c) + P(A|B)P(B)]\,P(B) \\ &= [P(A|B)P(B^c) + P(A|B)P(B)]\,P(B) \\ &= P(A|B)[P(B^c) + P(B)]P(B) \\ &= P(A|B)P(B) = P(A \cap B) \end{aligned}$$
  Solution 2: (Yue)
  By problem statement, we know that
  $$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c)}{P(B^c)}.$$
  Applying the fact
  $$\frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2} \text{ if } b_1 + b_2 \neq 0,$$
  we know
  $$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c) + P(A \cap B)}{P(B^c) + P(B)} = P(A).$$
  Thus A and B are independent.
- (10 pts) Notice that two inequalities can be rewritten as
  $$P(A \cap C) > P(B \cap C),$$
  $$P(A \cap C^c) > P(B \cap C^c).$$
  therefore by
  $$P(A) = P(A \cap C) + P(A \cap C^c),$$
  $$P(B) = P(B \cap C) + P(B \cap C^c),$$
  we have $P(A) > P(B)$.

4. (20 pts) Let the r.v. X have probability density function (pdf)

$$f(x) = \frac{1}{2} c x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}(x \in [0,1]) + \frac{1}{2} \mathbb{1}(x \in [0,1]).$$

- Find c such that f is a valid pdf.
- Compute $E[X]$.

Now, let $\alpha = 1, \beta = 1$.

- compute $P\left(X \in \left[\frac{1}{2}, 1\right]\right)$.

Solution:

- (8 pts) We need to find c such that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

By integration, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2} c \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx + \frac{1}{2} \int_0^1 1dx.$$

By definition of Beta distribution, we know $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = B(\alpha, \beta)$. Also $\int_0^1 1dx = 1$. Then we have

$$\frac{1}{2} cB(\alpha, \beta) + \frac{1}{2} = 1.$$

Solving this equation yields $c = \frac{1}{B(\alpha,\beta)}$.

- (6 pts) By expectation formula of Beta distribution and uniform distribution, we know

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} x dx = \frac{\alpha}{\alpha+\beta},$$

$$\int_0^1 x dx = \frac{1}{2}.$$

Inserting these into expression of $\mathbb{E}[X]$ yields $\mathbb{E}[X] = \frac{\alpha}{2(\alpha+\beta)} + \frac{1}{4}$.

- (6 pts) When $\alpha = \beta = 1$, we can see that X is uniform distribution on $[0, 1]$. Then we immediately have $P\left(X \in \left[\frac{1}{2}, 1\right]\right) = \frac{1}{2}$.

Remark: if part (a) was not correct, but part (b) and (c) are incorrect only due to part (a), then full score for parts (b) and (c).

5. (20 pts) Consider a r.v. $X \in [a, b]$, where $0 < a < b$ and $a, b \in \mathbb{R}$. Let

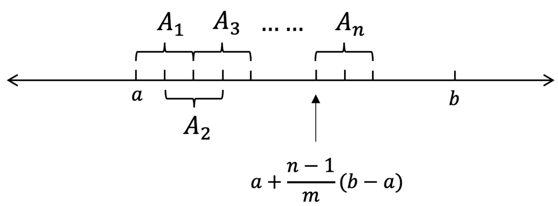
$$A_i = \left[a + \frac{(i-1)}{m}(b-a), a + \frac{(i+1)}{m}(b-a)\right]$$

for $i = 1, \ldots, n$ with $n < m - 1$. That is

$$A_1 = \left[a, a + \frac{2}{m}(b-a)\right], A_2 = \left[a + \frac{1}{m}(b-a), a + \frac{3(b-a)}{m}\right], \ldots$$

Now,

- let $P(X \in A_i) = \frac{c}{i}$, where c is some normalizing constant.[1] Find an upper bound for $P(X \in \cup_{i=1}^n A_i)$.
- let $E[X] = \sqrt{ab}$. Find an upper bound for $P(X \in \cup_{i=1}^n A_i)$.
- let $Y = \frac{X-a}{b-a}$ with $Y \sim B(\alpha, \beta)$ with $\alpha, \beta > 0$. Compute $P(X \in \cup_{i=1}^n A_i)$. You do not need - cannot - to find a closed form solution.



Solution:

- (10 pts) By union bound,

$$P(X \in \cup_{i=1}^n A_i) \le \sum_{i=1}^n P(A_i) = \sum_{i=1}^n \frac{c}{i}.$$

- (5 pts) By Markov inequality,

$$\begin{aligned} P(X \in \cup_{i=1}^n A_i) &= P(b - X \ge b - \left[a + \frac{n+1}{m}(b-a)\right]) \\ &\le \frac{\mathbb{E}[b-X]}{\frac{m-n-1}{m}(b-a)} = \frac{\sqrt{ab}}{\frac{m-n-1}{m}(b-a)} \end{aligned}$$

[1] If you are interested, we take $c = \left(\sum_{i=1}^m \frac{1}{i}\right)^{-1}$.

- (5 pts) We have $P(X \in \cup_{i=1}^{n} A_i) = P(a \leq X \leq \left[a + \frac{n+1}{m}(b-a)\right]) = P(0 \leq Y \leq \frac{n+1}{m})$. Thus the result is $\int_0^{\frac{n+1}{m}} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$.

Remark: for part (b), some methods will yield non-informative bounds, i.e., $P(X \in \cup_{i=1}^{n} A_i) \leq a$, where $a \geq 1$. As long as the reasoning is logically correct, you can get 2 to 3 points.

6. (20 pts) Two gamblers, Yue and Riccardo, bet 1$ each on the successive tosses of a coin. Each has a bank of 6$. What is the probability that

   - they break even after six tosses of the coin?
   - Yue wins all the money on the tenth toss of the coin?

   Solution:

   - (10 pts) To make them break even after 6 tosses, Riccardo needs to win 3 times and lose 3 times in the first 6 tosses. Thus the answer is

   $$\binom{6}{3}\left(\frac{1}{2}\right)^6.$$

   - (10 pts) The final answer is

   $$27\left(\frac{1}{2}\right)^{10}.$$

     First we think about what conditions are needed for Yue to win on the 10th toss. She needs to (1)win 8 out of the first 10 tosses; (2)win on 9th and 10th toss; (3)cannot win all the first 6 tosses. (You should check this in two ways: if these 3 conditions are satisfied, Yue will win on 10th toss; if she wins on 10th toss, these 3 conditions are satisfied.)

     To meet condition (1) and (2), Yue losses 2 out of the first 6 tosses, and wins others in the first 10 tosses. To meet condition (3), we need to exclude the result WWWWWWLLWW. The we have the final result:

   $$\left(\binom{8}{2}-1\right)\left(\frac{1}{2}\right)^{10}.$$