Test II (Friday May 14th, time allowed: 80 minutes)

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Total: 120 points. Full score: 100 points.

• Problems by Riccardo. Solutions by Yue.•

1. Let the joint pdf of the r.v.'s (X, Y) be

$$f_{X,Y}(x,y) = \begin{cases} cyx^2 \text{ if } 0 < x < 5,\ 0 < y < x \\ 0 \text{ o/w} \end{cases}$$

- (5 pts) Compute the normalizing constant c.
- (5 pts) Compute f_X.
- (5 pts) Compute $P(X \in [2,5])$.
- (5 pts) Compute $f_{Y|X=x}$ for $0 < x < 5$.

Solution:

- (5 pts) We need normalizing constant c such that

  [$2pts$] $$\int_0^5 \int_0^x cyx^2 dydx = 1.$$

  By integration,

  [$4pts$] $$\int_0^5 \int_0^x cyx^2 dydx = \int_0^5 \frac{c}{2} x^4 dx = \frac{625}{2} c.$$

  Thus we have

  [$5pts$] $$c = \frac{2}{625}.$$

- (5 pts) The marginal distribution of X is calculated by

  [$2pts$] $$f_X(x) = \int_0^x f_{X,Y}(x,y)dy.$$

  By integration,

  [$4pts$] $$f_X(x) = \int_0^x f_{X,Y}(x,y)dy = \frac{c}{2} x^4.$$

  Plug in the value of c, we have

  [$5pts$] $$f_X(x) = \frac{1}{625} x^4, x \in (0,5).$$

- (5 pts) The desired probability is calculated by

  [3*pts*] $$\mathbb{P}(X \in [2,5]) = \int_2^5 f_X(x)dx.$$

  Plug in the result of last subproblem, we have

  [5*pts*] $$\mathbb{P}(X \in [2,5]) = \int_2^5 f_X(x)dx = \frac{1}{625}(5^5/5 - 2^5/5) = \frac{3093}{3125}.$$

- (5 pts) The conditional density is calculated by

  [3*pts*] $$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, x \in (0,5).$$

  Plug in the result of (b), we have

  [5*pts*] $$f_{Y|X=x}(y|x) = \frac{2y}{x^2}, 0 < x < 5, 0 < y < x.$$

2. (20 pts) Let $X_1 \sim \text{Uniform}(0,1)$ and $X_2 \sim \text{Uniform}(0,2)$.[1] Moreover, $X_1 \perp\!\!\!\perp X_2$. Compute $F_Y(y)$ where $Y = X_1 - X_2$ and $y \in \mathbb{R}$.

   Solution: The cdf for $Y = X_1 - X_2$ can be written as

   $$[5pts] \quad F_Y(y) = \mathbb{P}\left(X_1 - X_2 \le y\right) = \mathbb{E}\left[\mathbb{P}\left(X_1 \le y + X_2 | X_2 = x_2\right)\right].$$

   We have

   $$[10pts] \qquad \mathbb{P}\left(X_1 \le y + x_2\right) = \min\left\{1, y + x_2\right\}.$$

   Thus the above expression is

   $$[+2pts \text{ each}] \qquad F_Y(y) = \begin{cases} 0, & y \le -2; \\ \frac{(y+2)^2}{4}, & -2 < y \le -1; \\ \frac{2y+3}{4}, & -1 < y \le 0; \\ 1 - \frac{(1-y)^2}{4}, & 0 < y \le 1; \\ 1, & y > 1. \end{cases}$$

[1] Pay attention to the different supports!

3. (20 pts) Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. Consider the realizations of those r.v.'s, that is $X_1 = x_1, \ldots, X_n = x_n$.

- (5 pts) Find the MLE for μ and the MLE for σ^2.
- (10 pts) Prove that the MLE estimator for σ^2 is biased.[2]
- (5 pts) Finally, find the MLE for $\sigma = \sqrt{\sigma^2}$.[3]

Solution:

- (5 pts) By lecture notes, the MLE for μ is

$$[2pts] \qquad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

  Derivation of the MLE for σ^2 goes like

$$[4pts] \qquad \frac{\partial}{\partial \sigma^2} \sum_{i=1}^{n} \left[-\frac{1}{2}\log 2\pi\sigma^2 - \frac{(x_i - \mu)^2}{2\sigma^2} \right] = 0.$$

  Then we have

$$[5pts] \qquad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \hat{\mu}^2.$$

- (10 pts) The bias of $\widehat{\sigma^2}$ is calculated by

$$\begin{aligned} \mathbb{E}[\widehat{\sigma^2}] - \sigma^2 =& \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i^2] - \mathbb{E}[\hat{\mu}^2] - \sigma^2 \\ =& (\mu^2 + \sigma^2) - (\mu^2 + \frac{1}{n}\sigma^2) - \sigma^2 \\ =& -\frac{1}{n}\sigma^2 < 0. \end{aligned}$$

- (5 pts) By invariance property of MLE, we immediately know

$$[5pts] \qquad \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2}.$$

[2] In order to prove this fact, you need to replace μ in the formula of $\hat{\sigma}^2$ with $\hat{\mu}$, its MLE. Moreover, it would be useful to remember that $V(Y) = E[Y^2] - (E[Y])^2$.

[3] Remember the properties of the MLE.

4. (20 pts) Let $X_1 \sim \text{Beta}(\alpha_1, \beta_1)$, $X_2 \sim \text{Bernoulli}(X_3)$ where $X_3 \sim \text{Beta}(\alpha_3, \beta_3)$. You also know that $X_1 \perp\!\!\!\perp (X_2, X_3)$.

   - (5 pts) Compute $E[X_1]$.[4]
   - (5 pts) Compute $E[X_2|X_3]$.
   - (5 pts) Compute $E[X_1 X_2]$.[5]
   - (5 pts) Compute $V(X_1 + X_2)$.[6]

   Solution:

   - (5 pts) We have

     $$\mathbb{E}[X_1] = \int_0^1 \frac{1}{B(\alpha_1, \beta_1)} x^{\alpha_1+1-1}(1-x)^{\beta_1-1} dx = \frac{B(\alpha_1+1, \beta_1)}{B(\alpha_1, \beta_1)} = \frac{\alpha_1}{\alpha_1+\beta_1}.$$

   - (5 pts) We have

     $$\mathbb{E}[X_2|X_3] = X_3.$$

   - (5 pts) By $X_1 \perp\!\!\!\perp (X_2, X_3)$, we have

     $$[2pts] \qquad X_1 \perp\!\!\!\perp X_2.$$

     $$[3pts] \qquad \mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2].$$

     We also have

     $$[5pts] \qquad \mathbb{E}[X_2] = \mathbb{E}[\mathbb{E}[X_2|X_3]] = \mathbb{E}[X_3] = \frac{\alpha_3}{\alpha_3+\beta_3}.$$

     Thus

     $$\mathbb{E}[X_1 X_2] = \frac{\alpha_1}{\alpha_1+\beta_1}\frac{\alpha_3}{\alpha_3+\beta_3}.$$

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[4] Since you can find the solution in the notes, I already help you with it: $E[X_1] = \frac{\alpha_1}{\alpha_1+\beta_1}$. However, you still need to prove it!

[5] You might need the following implication (proved in hw 5):

$$X_1 \perp\!\!\!\perp (X_2, X_3) \implies \begin{cases} X_1 \perp\!\!\!\perp X_2 | X_3 \\ X_1 \perp\!\!\!\perp X_3 | X_2 \end{cases}$$

[6] These computations might get quite messy (but not difficult), so leave them at the end. As long as you break down the variance correctly, and are able to rewrite everything in terms of the integral, you will get full score. Again, you can find the result in the notes so I'll help you with that: $V(X_1) = \frac{\alpha_1\beta_1}{(\alpha_1+\beta_1)^2(\alpha_1+\beta_1+1)}$.

- (5 pts) By $X_1 \perp\!\!\!\perp X_2$, we have

  [1pts] $$V(X_1 + X_2) = V(X_1) + V(X_2).$$

  Also we have

  [3pts] $$V(X_1) = \frac{\alpha_1\beta_1}{(\alpha_1 + \beta_1)^2(\alpha_1 + \beta_1 + 1)},$$

  $$\begin{aligned}
  V(X_2) =& V(\mathbb{E}[X_2|X_3]) + \mathbb{E}[V(X_2|X_3)] = V(X_3) + \mathbb{E}[X_3(1 - X_3)] \\
  =& V(X_3) + \mathbb{E}[X_3] - \mathbb{E}[X_3^2] \\
  =& \mathbb{E}[X_3]^2 + \mathbb{E}[X_3],
  \end{aligned}$$

  and thus

  [5pts] $$V(X_2) = \frac{\alpha_3}{\alpha_3 + \beta_3} + \left(\frac{\alpha_3}{\alpha_3 + \beta_3}\right)^2.$$

5. (20 pts) Let $Y = \exp\{\mu + \sigma Z\}$ where $Z \sim \mathcal{N}(0,1)$ and $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$.

- (10 pts) Show that[7]

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}}.$$

- (5 pts) Compute $E[Y]$.[8]
- (5 pts) Compute $V(Y)$.

Solution:

- Using the method of the cumulative distribution function, we have

$$\begin{aligned} f_Y(y) &= \frac{\partial P(Y \leq y)}{\partial y} = \frac{\partial}{\partial y} P\left(Z \leq \frac{\log(y) - \mu}{\sigma}\right) \\ &= \frac{\partial}{\partial y}\phi\left(\frac{\log(y) - \mu}{\sigma}\right) = f_Z\left(\frac{\log(y) - \mu}{\sigma}\right)\frac{1}{\sigma y}\mathbb{1}(y > 0) \\ &= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \mathbb{1}(y > 0). \end{aligned}$$

Using the method of the change of variable, we have

$$\begin{aligned} f_Y(y) &= f_Z\left(\frac{\log(y) - \mu}{\sigma}\right) \cdot \frac{\partial}{\partial y}\frac{\log y - \mu}{\sigma}\mathbb{1}(y > 0) \\ &= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \mathbb{1}(y > 0). \end{aligned}$$

- By hint, the expectation of Y is calculated by

$$\mathbb{E}[Y] = \mathbb{E}[e^{\mu} e^{\sigma Z}] = e^{\mu}\mathbb{E}[e^{\sigma Z}] = e^{\mu} e^{\sigma^2/2} = e^{\mu + \sigma^2/2}.$$

[7] You *might* need the following fact from the previous homework:

$$\frac{\partial \phi(g(y))}{\partial y} = f_Z(g(y))\frac{\partial g(y)}{\partial y}.$$

[8] You *will need* the following fact, shown in the homework,$E[e^{tZ}] = e^{\frac{t^2}{2}}$. Do not try to integrate Y. Instead, use linearity of the integral togerther with this fact.

- First we have

$$[3pts] \quad \mathbb{E}[Y^2] = \mathbb{E}[e^{2\mu+2\sigma Z}] = e^{2\mu}\mathbb{E}[e^{2\sigma Z}] = e^{2\mu}e^{2\sigma^2} = e^{2\mu+2\sigma^2}.$$

  Then,
  [5pts]
  $V[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$

6. Let the r.v. X have pdf

$$f_X(x) = \frac{1}{10}\mathbb{1}(x \in [0, 10]).$$

Now let $Y|X \sim \text{Bernoulli}(m^*(X))$ where $m^*(x) = x/(x+1)$. Moreover, let $h(X) = \mathbb{1}(m(X) > 1/2)$ where $m(X) = x/(x+2)$.

- (5 pts) Compute $E[Y]$ and $E[h(X)]$.[9]
- (5 pts) Compute $E[h(X)|Y = 0]$. [10]
- (5 pts) Compute $E[(h(X) - Y)^2]$.[11]
- (5 pts) Compute $P(X \in [2, 6]|h(X) = 1)$.

Solution:

- (5 pts) First we have

$$[2pts] \qquad \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[m^*(X)] = \mathbb{E}[\frac{x}{x+1}],$$

  [3pts]

$$\mathbb{E}[\frac{x}{x+1}] = \frac{1}{10}\int_0^{10} \frac{x}{x+1}dx = \frac{1}{10}\left[x - \ln(x+1)\right]\Big|_0^{10} = \frac{10 - \ln 11}{10}.$$

  The function $h(x)$ can be written as $h(x) = \mathbb{1}(x > 2)$, and thus

$$[5pts] \qquad \mathbb{E}[h(X)] = \mathbb{E}[\mathbb{1}(X > 2)] = \mathbb{P}(X > 2) = \frac{4}{5}.$$

[9] You will need the following fact: $\int \frac{x}{c+x}dx = x - c\log(x+c)$ where c is some constant larger than 0, and $x > 0$.

[10] Remember that $E[h(X)] = P(h(X) = 1)$ for $h \in \{0, 1\}$ and Bayes theorem might be useful here.

[11] You might want to use linearity of the integral after opening the square.

- (5 pts) First we have

  $$[2pts] \quad \mathbb{E}[h(X)|Y=0] = \mathbb{P}(X>2|Y=0) = \frac{\mathbb{P}(X>2, Y=0)}{\mathbb{P}(Y=0)}.$$

  To calculate this, we need
  [4pts]
  $$\mathbb{P}(X>2, Y=0) = \int_2^{10} f_X(x)\mathbb{P}(Y=0|X=x)dx = \int_2^{10} \frac{1}{10}\frac{1}{x+1}dx = \frac{\ln(11/2)}{10},$$

  and

  $$[5pts] \qquad \mathbb{P}(Y=0) = \int_0^{10} \frac{1}{10}\frac{1}{x+1}dx = \frac{\ln 11}{10}.$$

- (5 pts) Because we have $h^2(x) = h(x)$ and $Y^2 = Y$, we can rewrite the expectation as

  $$[2pts] \ \mathbb{E}[h^2(X) - 2Yh(X) + Y^2] = \mathbb{E}[h(X)] - 2\mathbb{E}[Yh(X)] + \mathbb{E}[Y].$$

  Note that

  $$\mathbb{E}[Yh(X)] = \mathbb{P}(Y=1)\mathbb{E}[h(X)|Y=1],$$

  and similar to the last problem, we have

  $$[3pts] \qquad \mathbb{E}[h(X)|Y=1] = \frac{\mathbb{P}(X>2, Y=1)}{\mathbb{P}(Y=1)} = \frac{8-\ln(11/2)}{10-\ln 11}.$$

  Also we have $\mathbb{P}(Y=1) = \mathbb{E}[Y]$. Thus the final expression would be

  $$[5pts] \qquad E[(h(X)-Y)^2] = \frac{2-\ln(11/4)}{10}.$$

- (5 pts) The desired expression is

  $$[5pts] \ \ P(X\in[2,6]|h(X)=1) = P(X\in[2,6]|X\geq 2) = \frac{4}{8} = \frac{1}{2}.$$