[36-217] - 06/06/2019

(Spoiler: the document may contain -many- mistakes)

INTEGRALS (RIEMANN)

LET $f:[a,b]\to\mathbb{R}$. CONSIDER THE PARTITION Π OF $[a,b]$:

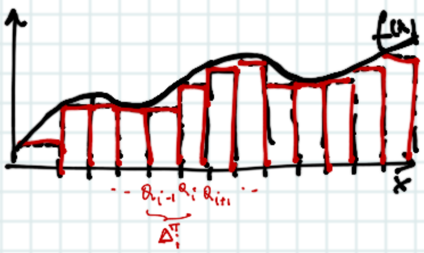
$$\Pi=\{\pi=(a_1,\dots,a_N):\ a=a_1<\dots<a_N=b,\ N\in\mathbb{N}\}.$$

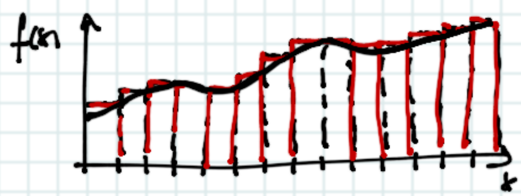
FOR EVERY $\pi\in\Pi$ CONSIDER, FOR $i=1,\dots,N$,

$$m_i^\pi=\inf_{x\in[a_{i-1},a_i]} f(x) \qquad M_i^\pi=\sup_{x_i\in(a_{i-1},a_i]} f(x)$$

AND LET $\Delta_i^\pi=a_i-a_{i-1}$. THE LOWER AND UPPER RIEMANN SUMS ARE DEFINED AS FOLLOWS:

$$L^\pi(f)=\sum_{i=1}^N m_i^\pi\Delta_i^\pi \qquad U^\pi(f)=\sum_{i=1}^N M_i^\pi\Delta_i^\pi$$





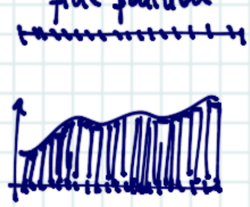
THE LOWER AND UPPER INTEGRAL ARE DEFINED AS

$$\underline{\int_a^b} f(x)\,dx=\sup_{\pi\in\Pi}\sum_{i=1}^N m_i^\pi\Delta_i \qquad \overline{\int_a^b} f(x)\,dx=\inf_{\pi\in\Pi}\sum_{i=1}^N M_i^\pi\Delta_i$$

Intuition: you can think about these two integrals as taking a fine partition, that is

coarse partition

fine partition



(lower sum) ~bad approx ~good approx!

IF THE LOWER AND THE UPPER INTEGRALS HAVE THE SAME VALUE, THEN f IS SAID TO BE <u>RIEMANN INTEGRABLE</u>, AND WE WRITE

$$\int_a^b f(x)\,dx=\underline{\int_a^b} f(x)\,dx=\overline{\int_a^b} f(x)\,dx$$

• WHEN IS A FUNCTION NOT RIEMANN INTEGRABLE?

THE GENERAL ANSWER IS THE FOLLOWING STATEMENT:

A FUNCTION $f: [a,b] \to \mathbb{R}$ IS RIEMANN INTEGRABLE IF AND ONLY IF IT IS BOUNDED AND ITS SET OF POINTS OF DISCONTINUITY HAS LEBESGUE MEASURE ZERO.

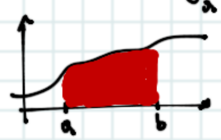
HOWEVER, THE STATEMENT INVOLVES LEBESGUE MEASURES, THAT WE HAVE NOT COVERED.

IF YOU'RE INTERESTED, AN EXAMPLE OF A FUNCTION THAT IS NOT RIEMANN INTEGRABLE IS

$$f(x) = \mathbb{1}_{[a,b]\cap\mathbb{Q}}(x) = \begin{cases} 1 & \text{IF } x \in [a,b]\cap\mathbb{Q} \\ 0 & \text{O/W} \end{cases}$$

($\mathbb{Q}$: RATIONAL NUMBERS) THAT IS CALLED THE DIRICHLET FUNCTION.

• TO CONCLUDE, WE MAY INFORMALLY SAY THAT $\int_a^b f(x)\,dx$ IS MEASURING THE AREA UNDER f:

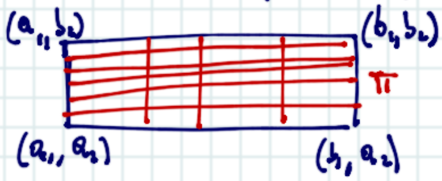


– MULTIPLE INTEGRALS

I'LL BE A BIT MORE INFORMAL FROM NOW ON.

LET $f: [a_1,b_1]\times[a_2,b_2]\times \dots \times [a_m,b_m] \to \mathbb{R}$. AGAIN WE LET Π BE THE SET OF PARTITIONS OF THE m-th DIMENSIONAL SPACE.

For the case of $[a_1,b_1]\times[a_2,b_2]$ you can imagine one of the partitions to be as in the following:



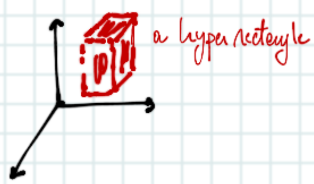
You can see that $\pi = \pi_1 \times \dots \times \pi_m$, that is the partition can be rewritten as the Cartesian product of the 1-dim. partitions.

CALL $B_1 \dots B_N$ THE SETS FORMING THE PARTITION π. AGAIN, IF THE LOWER AND UPPER RIEMANN SUMS COINCIDE:

$$L^{\pi}(f) = \sum_{i=1}^{N} m_i^{\pi} \Delta_i^{\pi} \qquad U^{\pi}(f) = \sum_{i=1}^{N} M_i^{\pi} \Delta_i^{\pi}$$

[WHERE $\Delta_i^{\pi} = (a_{1i} - a_{1i-1})(a_{2i} - a_{2i-1}) \cdots (a_{mi} - a_{mi-1})$ (IE THE VOLUME OF THE HYPERRECTANGLE) AND m_i^{π}, M_i^{π} ARE DEFINED IN A SIMILAR WAY AS BEFORE]

THEN f IS SAID TO BE RIEMANN INTEGRABLE.



Intuition: we may write that

$$\int_{[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_m,b_m]} f(x)\,dx \approx \sum_{i=1}^{N} f(\underbrace{z_i}_{\text{point in } B_i})\,\underbrace{\Delta_i}_{\text{volume of } B_i \text{ (hyperrectangle)}}$$

• NOW WE WILL JUST WORK WITH THE CASE OF $n=2$ TO SIMPLIFY NOTATION. HOWEVER, EVERYTHING IS STILL VALID IN THE MORE GENERAL CASE OF $n>2$.

• LET $f:[a_1,b_1]\times[a_2,b_2]\to\mathbb{R}$, AN $A = A_1\times A_2 = [a_1',b_1']\times[a_2',b_2'] \subseteq [a_1,b_1]\times[a_2,b_2]$. WE WRITE THE INTEGRAL AS

$$\int_A f(x,y)\, d(x,y) \qquad \text{OR} \qquad \int_{A_1}\int_{A_2} f(x_1,x_2)\, dx_2\, dx_1$$

Intuition: $\int_{A_1}\int_{A_2} f(x_1,x_2)\,dx_2\,dx_1$ can be thought as "fix x_1 and integrate over x_2" and then "integrate that value over x_1"

PROPERTIES:

(1) $\iint [f(x,y)+g(x,y)]\,dx\,dy = \iint f(x,y)\,dx\,dy + \iint g(x,y)\,dx\,dy$

(2) $\iint k\, f(x,y)\,dx\,dy = k\iint k f(x,y)\,dx\,dy$ for $k\in\mathbb{R}$

(3) $f(x,y)\le g(x,y) \Rightarrow \iint f(x,y)\,dx\,dy \le \iint g(x,y)\,dx\,dy$

(4) $\iint_{A\cup B} f(x,y)\,dx\,dy = \iint_A f(x,y)\,dx\,dy + \iint_B f(x,y)\,dx\,dy$

(...) LOOK ONLINE OR PROVE THEM!

– SOME VERY USEFUL PROPERTIES/RULES OF INTEGRALS (not correctly ordered)

(1) LEIBNIZ INTEGRATION RULE (hp: f and partial derivatives need to be continuous)

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\frac{\partial}{\partial x}b(x) - f(x,a(x))\frac{\partial}{\partial x}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt$$

(2) FUNDAMENTAL THEOREM OF CALCULUS (corollary of Leibniz rule)

$$\frac{\partial}{\partial x}\int_a^x f(t)\,dt = f(x)$$

[this is why $\frac{\partial \overset{CDF}{F_X(x)}}{\partial x} = \overset{PDF}{f(x)}$!]

(3) FUBINI-TONELLI THEOREM (A_1 and A_2 need to be σ-finite measure spaces)

$$\int_{A_1}\int_{A_2} |f(x,y)|\,dy\,dx = \int_{A_2}\int_{A_1} |f(x,y)|\,dx\,dy = \int_A |f(x,y)|\,d(x,y)$$

SOME EXERCISES:

(1) $\iint_A (2x-3y^2)\,dx\,dy$ $\qquad$ $A: -1 \le x \le 1,\ 0 \le y \le 2$

Solution: $\int_0^2\int_{-1}^1 (2x-3y^2)\,dx\,dy \overset{\text{Linearity}}{=} \int_0^2\int_{-1}^1 2x\,dx\,dy - \underbrace{\int_0^2\int_{-1}^1 3y^2\,dx\,dy}$

$$\int_0^2 3y^2 \int_{-1}^1 dx\,dy = \int_0^2 3y^2\ x\Big|_{-1}^1 dy = \int_0^2 6y^2\,dy = 6\,\frac{y^3}{3}\Big|_0^2 = 16$$

$$= \int_0^2 \underbrace{x^2\Big|_{-1}^1}_{1-1=0} dy - 16 = -16$$

(2) $\iint_A \frac{x}{y}\,dx\,dy$ $\qquad$ $A: 0 \le x \le 2,\ 1 \le y \le e$

Solution: $\int_1^e\int_0^2 \frac{x}{y}\,dx\,dy = \int_1^e \frac{1}{y}\int_0^2 x\,dx\,dy = 2\int_1^e \frac{1}{y}\,dy = 2(\log y)\Big|_1^e = 2$

(3) $\iint_A xy\,dx\,dy$ $\qquad$ $A: x^2+y^2 \le 25,\ x \ge 0,\ y \ge 0$

Solution: We have

remember that $\mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,\mathbb{1}(x\ge 0, y\ge 0,\ x^2 \le 25-y^2)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\ \mathbb{1}(x\ge 0)\mathbb{1}(y\ge 0)\mathbb{1}(x^2\le 25-y^2)\,dx\,dy = \int_0^{+\infty}\int_0^{+\infty} xy\,\mathbb{1}(x^2\le 25-y^2)\,dx\,dy$$

$$= \int_0^5 \int_0^{\sqrt{25-y^2}} xy\,dx\,dy =$$

Variation: $A: x+y \le 25,\ x \ge 0,\ y \ge 0$

Solution: Now,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\ \mathbb{1}(x\ge 0)\mathbb{1}(y\ge 0)\mathbb{1}(x+y\le 5)\,dx\,dy = \int_0^{25}\int_0^{25-y} xy\,dx\,dy$$

Variation: $A: x-y \le 25,\ x \ge 0$

Solution: $\int_{25}^{+\infty}\int_0^{25+y} xy\,dx\,dy$

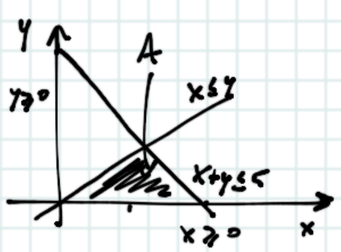
- IN GENERAL, WHAT'S THE "BEST" WAY TO FIND THE SUPPORT OF (X,Y) R.V.'S?

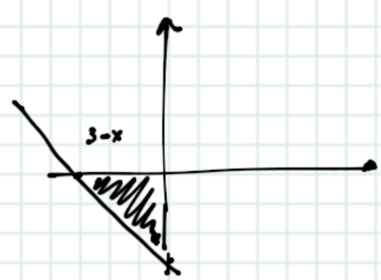
(1) DRAW IT $\qquad$ (2) REARRANGE THE INDICATOR FUNCTIONS

Q EXERCISES:

$\text{supp}((X,Y)) = A \cup B$ where $A = \begin{cases} x \ge 0 \\ y \ge 0 \\ x+y \le 5 \\ x \ge y \end{cases}$ and $\qquad B = $ and $\begin{cases} y \le 0 \\ y \ge -3-x \\ x \le 0 \end{cases}$

(1)





hence $$\underbrace{\int_0^{5/2}\int_0^{x} f(x,y)\,dy\,dx + \int_{5/2}^{5}\int_0^{5-y} f(x,y)\,dy\,dx}_{\int_A f(x,y)\,d(x,y)} + \underbrace{\int_{-3}^{0}\int_{-3-x}^{0} f(x,y)\,dy\,dx}_{\int_B f(x,y)\,d(x,y)}$$

equivalently, $$\overbrace{\int_0^{5/2}\int_y^{5-y} f(x,y)\,dx\,dy}\qquad + \overbrace{\int_{-3}^{0}\int_{-3-y}^{0} f(x,y)\,dx\,dy}$$

(2) $$\int_{A\cup B} f(x,y)\,d(x,y) = \underbrace{\int_A f(x,y)\,d(x,y)}_{①} + \underbrace{\int_B f(x,y)\,d(x,y)}_{②}$$

$$① = \iint_{-\infty}^{+\infty} f(x,y)\,\mathbb{1}(x\geq 0, y\geq 0, x+y\leq 5, x\geq y)\,dx\,dy = \iint_{-\infty}^{+\infty} f(x,y)\,\mathbb{1}(x\geq 0)\,\mathbb{1}(y\geq 0)\,\mathbb{1}(x+y\leq 5)\,\mathbb{1}(x\geq y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\mathbb{1}(y\geq 0)\int_{-\infty}^{+\infty} f(x,y)\,\mathbb{1}(x\geq 0)\,\mathbb{1}(x\leq 5-y)\,\mathbb{1}(x\geq y)\,dx\,dy = \int_0^{+\infty}\int_0^{+\infty} f(x,y)\,\mathbb{1}(x\leq 5-y)\,\mathbb{1}(x\geq y)\,dx\,dy$$

$$= \int_0^{5/2}\int_y^{5-y} f(x,y)\,dx\,dy$$

$$② = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\mathbb{1}(x\leq 0, y\geq -3-x, y\leq 0)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\overset{\mathbb{1}(y\leq 0)}{\mathbb{1}(x\leq 0)}\,\mathbb{1}(y\geq -3-x)\,dx\,dy = \int_{-\infty}^{0}\int_{-\infty}^{0} f(x,y)\,\mathbb{1}(x\geq -3-y)\,dx\,dy$$

$$= \int_{-3}^{0}\int_{-3-y}^{0} f(x,y)\,dx\,dy$$

Similarly, for the first case we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dy\,dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\underbrace{\mathbb{1}(x\geq 0)\,\mathbb{1}(y\geq 0)\,\mathbb{1}(x\geq y)\,\mathbb{1}(x+y\leq 5)}_{\Rightarrow\ \mathbb{1}(y\leq 5/2)!}\,dy\,dx = \int_0^{+\infty}\int_0^{+\infty} f(x,y)\,\mathbb{1}(x\geq y)\,\mathbb{1}(y\leq 5-x)\,dy\,dx$$

$$= \int_0^{5}\int_0^{\min\{x,5-x\}} f(x,y)\,dy\,dx = \int_0^{5/2}\int_0^{x} f(x,y)\,dy\,dx + \int_{5/2}^{5}\int_0^{5-x} f(x,y)\,dy\,dx$$

(Solve the last case by yourself)

b

WHE HAVE CLAIMED IN CLASS THAT

$$P(X+Y\leq c) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\mathbb{1}_A(x,y)\,dx\,dy \qquad \text{for } A\subseteq\mathbb{R}$$

BUT EXACTLY WHY IS THIS THE CASE?

NOW WE PROVE THIS (TRIVIAL) CLAIM:

$$P(X+Y\leq c) = \mathbb{E}_{X,Y}\left[\mathbb{1}(X+Y\leq c)\right] = \mathbb{E}_Y\left[\mathbb{E}_X\left[\mathbb{1}(X+Y\leq c)\,|\,Y=y\right]\right] = \int_{-\infty}^{+\infty}\mathbb{E}\left[\mathbb{1}(X+Y\leq c)\,|\,Y=y\right] f_Y(y)\,\mathbb{1}(y\in A_y)\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbb{1}(x+y\leq c)\,\underbrace{f_{X|Y=y}(x)\,\mathbb{1}(x\in A_{X|Y=y})\,f_Y(y)\,\mathbb{1}(y\in A_y)}_{f_{X,Y}(x,y)\,\mathbb{1}((x,y)\in A)}\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{c-y} f(x,y)\,\mathbb{1}((x,y)\in A)\,dx\,dy$$

Think carefully why $\mathbb{1}(x\in A_{x|Y})\mathbb{1}(y\in A_y) = \mathbb{1}((x,y)\in A)$!

Although the entire argument is trivial, make sure that you understand all the steps.