Convex Optimization
CMU-10725

3. Linear Programs

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Administrivia

- **Please ask questions!**

- Slides: http://www.stat.cmu.edu/~ryantibs/convexopt/

- Anonym feedback survey will be on black board today.
  Please use it! Constructive feedback and suggestions are always welcome!

- 1st recitation on Wednesday by Aaditya:
  - Linear algebra, Calculus, Probability
  - Linear Programming (another simplex method)

Basic Definitions

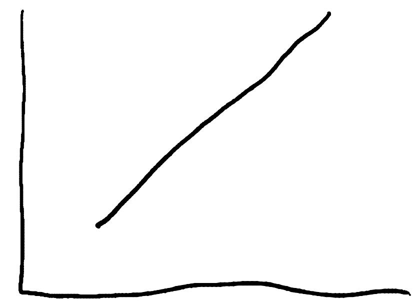
- More and more complicated optimization problems
- Definition of LP

Simplest Optimization Problems

Goal: $\min f(x)$ OR $\max f(x)$

- Constant function $f(x) = c \quad x \in \mathbb{R}^n$
- 1-dim linear function $f(x) = ax + b$

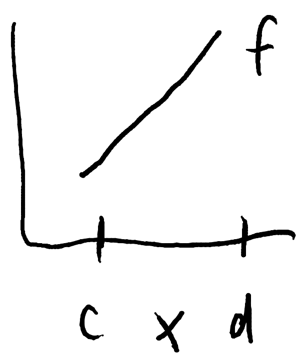
$\max_x f(x) = \infty \quad \min_x f(x) = -\infty$



- 1-dim linear function with bound constraints

$x \geq c \Rightarrow \operatorname{argmin} f(x) = c$ IF $a > 0$

$x \leq d \Rightarrow \operatorname{argmax} f(x) = d$ IF $a > 0$



Linear Programs

- n-dim linear function with m linear constraints

Inequality form:

Cost function: MIN [OR MAX] $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Constraints:

$$\text{S.T.} \quad a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \le b_1$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \le b_m$$

Bounds:

$$l_1 \le x_1 \le u_1$$
$$\vdots$$
$$l_m \le x_m \le u_m$$

l_i CAN BE $-\infty$

u_i CAN BE ∞

Linear Programs

Inequality form using matrix notation:

Cost function: MIN [OR MAX] $c^T x$ $\quad c \in \mathbb{R}^n$

Constraints: S.T $Ax \le b$ $\quad A \in \mathbb{R}^{m \times n}$ $\quad b \in \mathbb{R}^m$

Bounds: $l \le x \le u$ $\quad l \in \mathbb{R}^n$ $\quad u \in \mathbb{R}^n$

$x \in \mathbb{R}^n$

Example:

$$\begin{aligned} \text{MIN } & -2x_1 - x_2 \\ \text{S.T } & x_1 + x_2 \le 5 \\ & 2x_1 + 3x_2 \le 12 \\ & x_1 \le 4 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

$$c = [-2, -1]$$

$$A = \begin{bmatrix} 1, & 1 \\ 2, & 3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$l = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u = \begin{bmatrix} 4 \\ \infty \end{bmatrix}$$

Goal of this (...and next) lecture(s)

- ❑ **To be able to solve Linear Programs**

  Simplex Algorithm (Phase I and Phase II)
  (Later we will see other algorithms too)

- ❑ **Understand why LP is useful**
  - Motivation
  - Applications in Machine Learning

- ❑ **Understand the difficulties**
  - Convergence? Polynomial or Exponential many operations?
  - Will algorithms find the exact solutions, or only approximate ones?

Table of Contents

- **Motivating Examples & Applications:**
  - Pattern classification
- **Linear programs**:
  - standard form
  - canonical form
- **Solutions:**
  - Basic, Feasible, Optimal, Degenerate
- **Simplex algorithm:**
  - Phase I
  - Phase II

Linear Programs

- Motivation
- History
- Sketching LP

History

Dantzig 1947 (Simplex method)

(one of the top 10 algorithms of the twentieth century**)**

Motivated by World War II:

- ❑ Job scheduling (Assign 70 men to 70 jobs)
- ❑ Blending problem
  (produce a blend (30% Lead, 30% Zinc, 40% Tin) out of 9 different alloys (different mixture, different costs) such that the cost is minimal)
- ❑ Network flow optimization (Max flow min cut)

The product mix problem

A furniture company manufactures four models of desks

Number of man hours and profit:

	Desk 1	Desk 2	Desk 3	Desk 4	Available hrs
Carpentry shop hrs	4	9	7	10	6000
Finishing shop hrs	1	1	3	40	4000
Profit	$12	$20	$18	$40	

$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0$

MAX PROFIT $= 12x_1 + 20x_2 + 18x_3 + 40x_4$

S.T. $4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6000$

$x_1 + x_2 + 3x_3 + 40x_4 \leq 4000$

Why is it called Linear Programing???

Motivation: Why Linear Programing?

- The simplest, nontrivial optimization problem

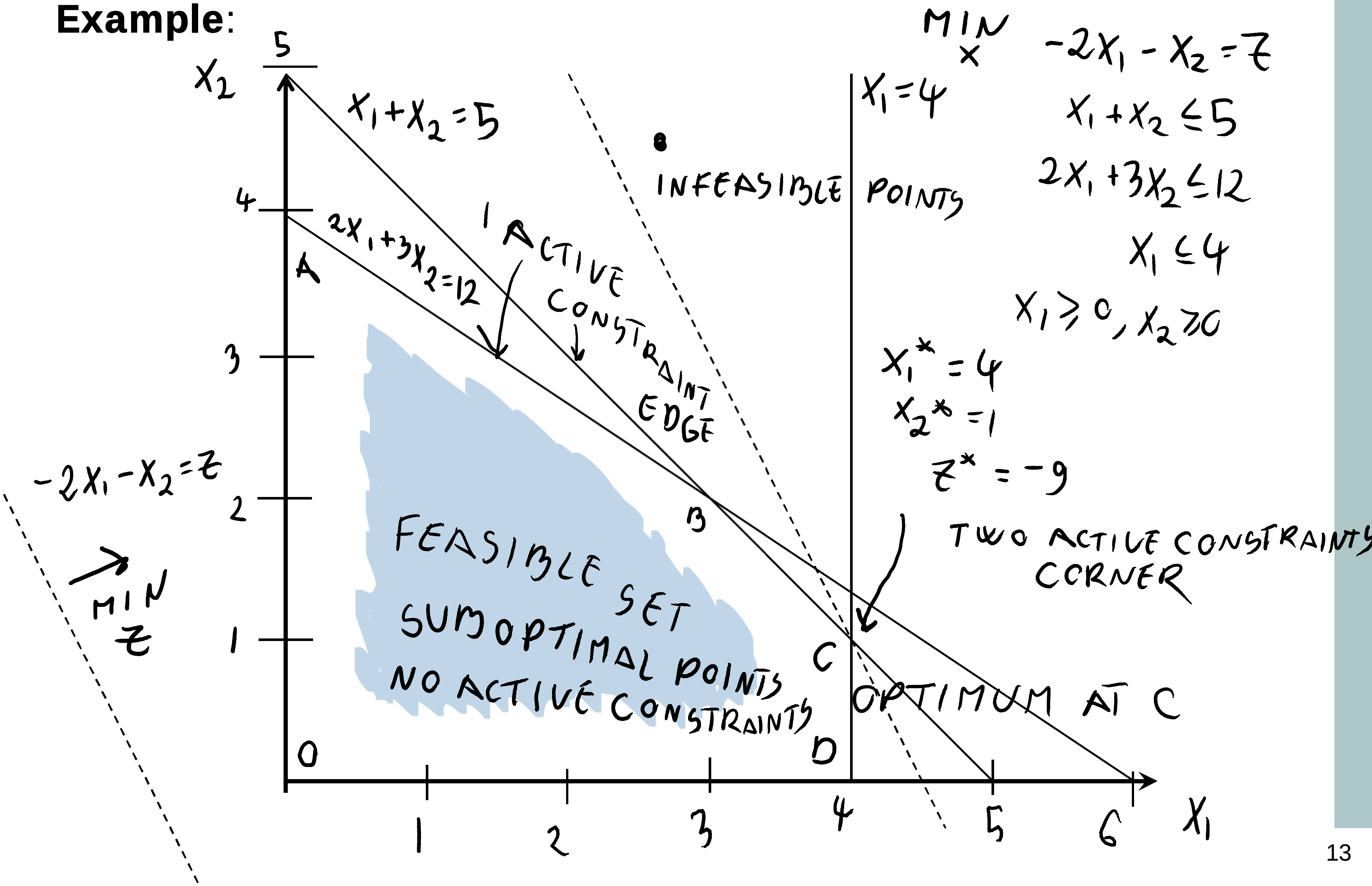
- Many complex system (objective and constraints) can be well approximated with linear equations

- Important applications

- There are efficient toolboxes that can solve LPs

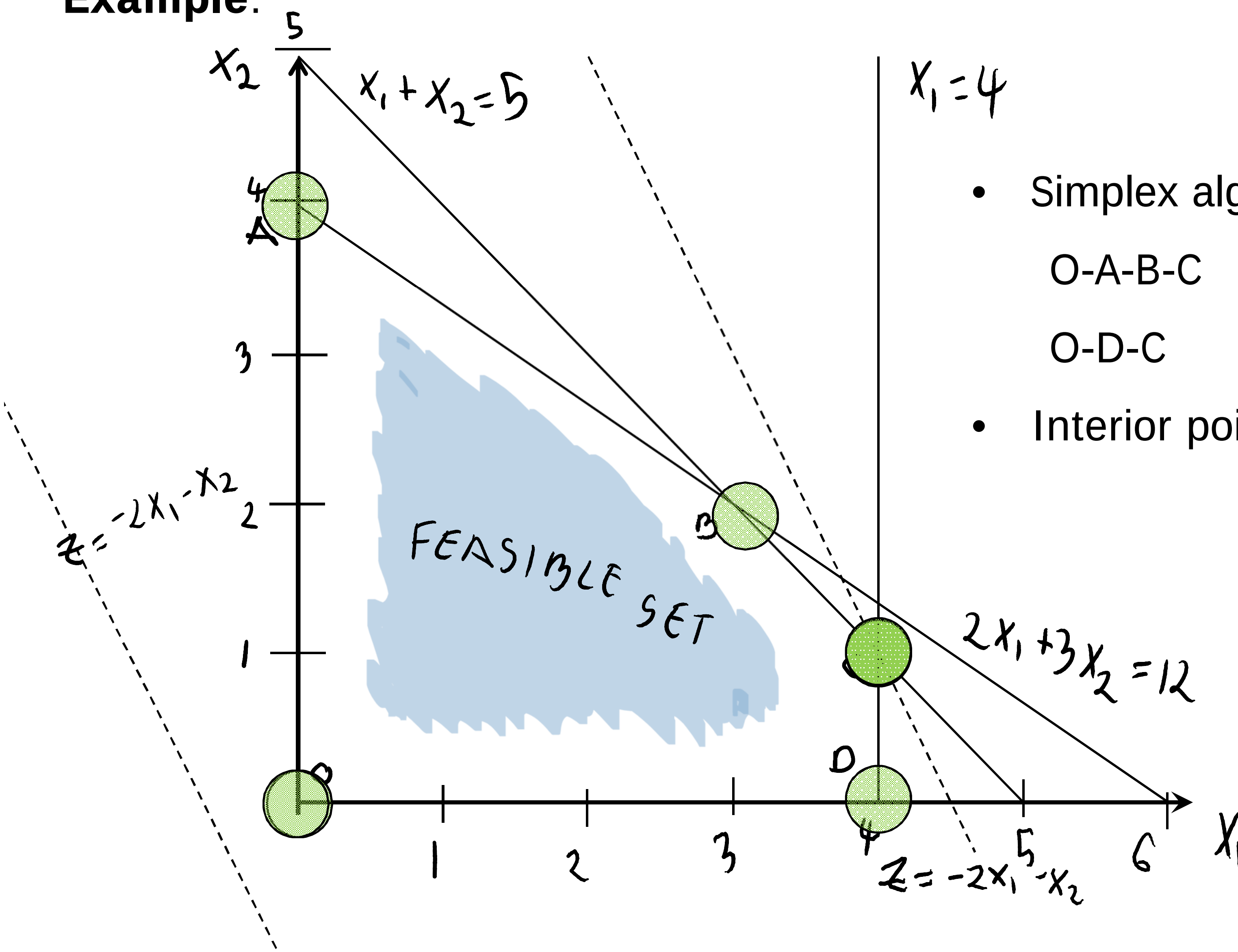
Sketching Linear Programs

Example:



Simplex Algorithm

Example:



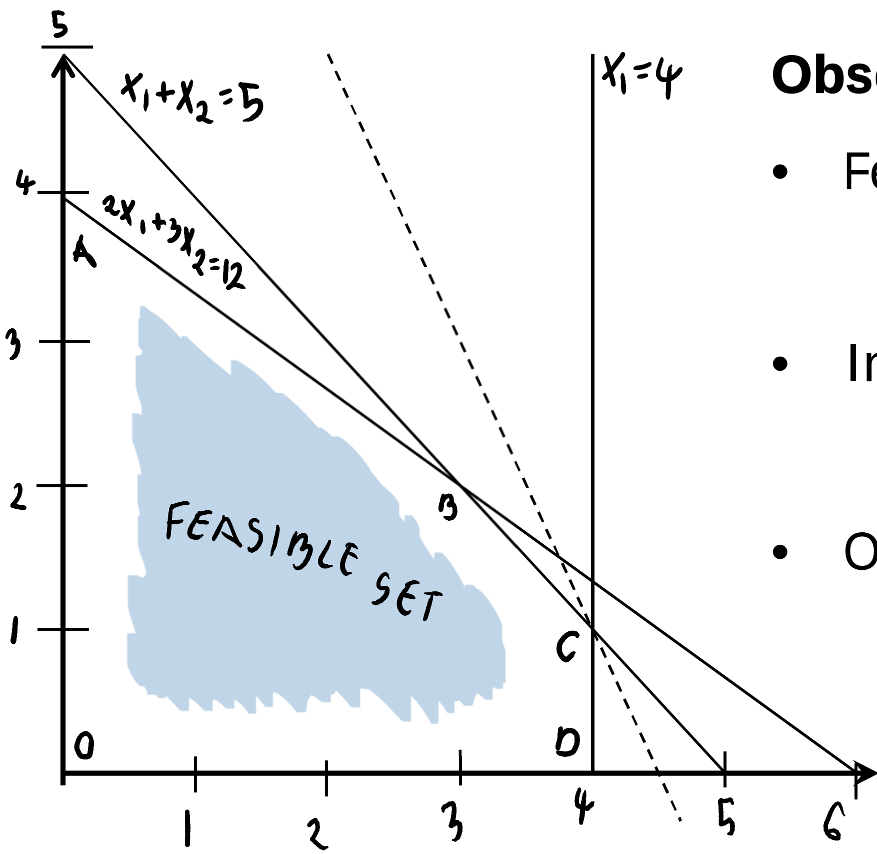
- Simplex algorithm:

  O-A-B-C

  O-D-C

- Interior point methods

Linear Program



Observations, Difficulties:

- Feasible set might not exist, no solution

  (Inconsistency in the constraints)

- Infinite many global optimum

  (Optimum is on an edge)

- Optimum can be – 1 , 1

  (Unbounded optimum)

$$\text{MIN [OR MAX] } c^T x \qquad c \in \mathbb{R}^n$$

$$\text{S.T. } Ax \leq b \qquad A \in \mathbb{R}^{m \times n}$$

$$\ell \leq x \leq u \qquad \ell, u, x \in \mathbb{R}^n$$

Linear Program

High dimensional case is similar:

faces, facets instead of edges

cost function = hyperplane

MIN [OR MAX] $c^T x$ $\quad c \in \mathbb{R}^n$

S.T. $Ax \leq b$ $\quad A \in \mathbb{R}^{m \times n}$

$l \leq x \leq u$ $\quad l, u, x \in \mathbb{R}^n$

Applications

Pattern Classification via Linear Programming

Application

Pattern Classification via Linear Programming

More info can be found on: cgm.cs.mcgill.ca/~beezer/cs644/main.html

Goal: show how LP can be used for linear classification.

Why LP?

There are many efficient LP solver software packages

Pattern Classification via LP

Formal goal:

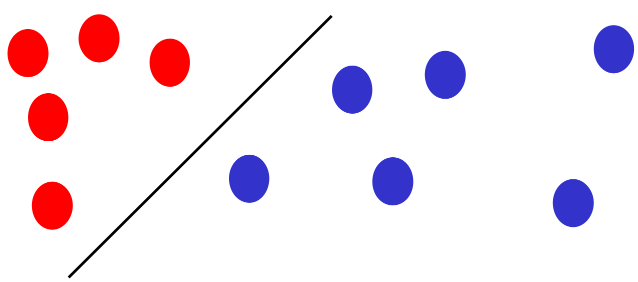
GIVEN $H=\{H^1,H^2,\dots,H^h\}\subseteq\mathbb{R}^n$ DISJOINT SETS

$M=\{M^1,M^2,\dots,M^m\}\subseteq\mathbb{R}^n$

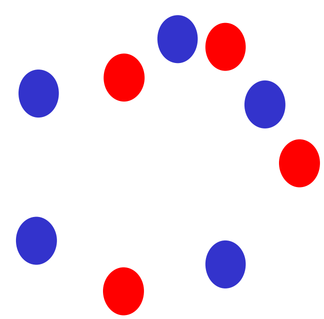
Problem 1: Determine whether H and M are linearly separable

Problem 2: If H and M are linearly separable,
then find a separating hyper plane

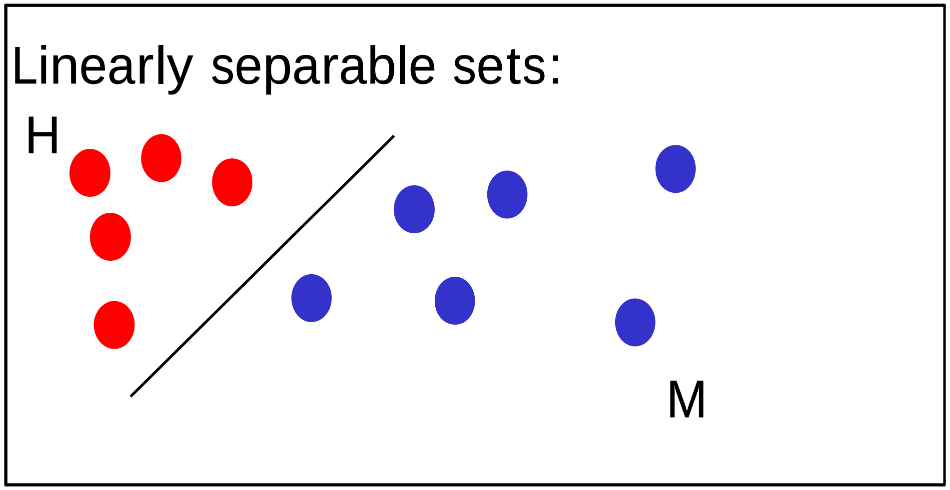
Linearly separable sets:

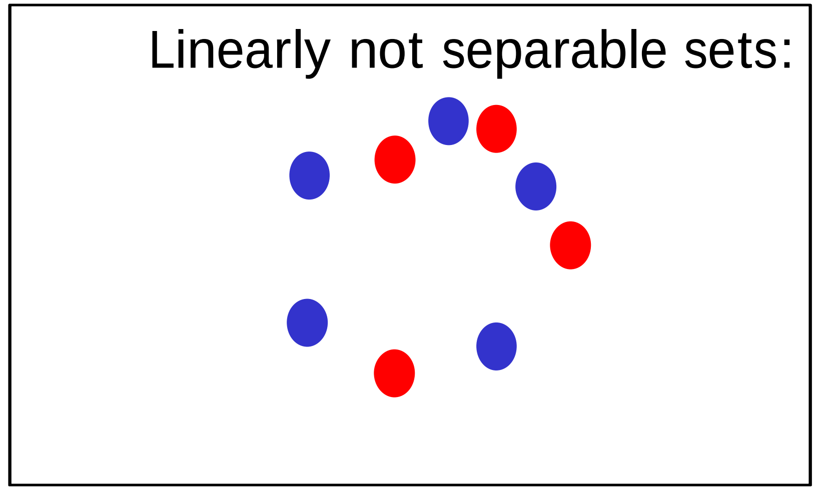


Linearly not separable sets:



Pattern Classification via LP





Observation:

H and M are linearly separable $\iff \exists\, a \in \mathbb{R}^n,\ b \in \mathbb{R}$ s.t. $H \subseteq \{x \in \mathbb{R}^n : a^T x > b\}$, $M \subseteq \{x \in \mathbb{R}^n : a^T x \leq b\}$

Pattern Classification via LP

Lemma 1:

$H = \{H^1, H^2, \ldots, H^h\} \subseteq \mathbb{R}^n$

$M = \{M^1, M^2, \ldots, M^m\} \subseteq \mathbb{R}^n$

H and M are linearly separable

$$\Leftrightarrow \exists \begin{matrix} a \in \mathbb{R}^n \\ b \in \mathbb{R} \end{matrix} \text{ s.t. } \begin{matrix} a^T H^i - b \geq +1 \quad \forall i = 1..h \\ a^T M^j - b \leq -1 \quad \forall j = 1..m \end{matrix}$$

Proof

$\Leftarrow$ TRIVIAL

$\Rightarrow$: SINCE $f(x) = a^T x - b$ SEPARATES H AND M

THEREFORE, $\{x : a^T x = b\}$ IS A SEPARATING HYPERPLANE

$\Rightarrow$ H AND M ARE LINEARLY SEPARABLE.

Pattern Classification via LP

Lemma 1:

H and M are linearly separable

$$\Longleftrightarrow \exists \; a \in \mathbb{R}^n, \; b \in \mathbb{R} \text{ s.t. } \begin{array}{l} a^T H^i - b \geq +1 \quad \forall i = 1..h \\ a^T M^j - b \leq -1 \quad \forall j = 1..m \end{array}$$

$\Rightarrow$ IS A LITTLE MORE COMPLICATED

Proof

$$\Rightarrow \exists \; c \in \mathbb{R}^n, \; b \in \mathbb{R} : \begin{array}{l} c^T x > b \quad \forall x \in H \\ \leq b \quad \forall x \in M \end{array}$$

$$\Rightarrow \min_{x \in H} c^T x > b \geq \max_{x \in M} c^T x$$

$$\Rightarrow \text{LET } \rho \doteq \min_{x \in H} c^T x - \max_{x \in M} c^T x > 0$$

Pattern Classification via LP

Proof continued

$$\Rightarrow \text{LET } \rho \doteq \min_{x \in H} c^T x - \max_{x \in M} c^T x > 0$$

$$\text{LET } a \doteq \frac{2}{\rho} c \in \mathbb{R}^n, \quad b \doteq \frac{1}{\rho}\left[\min_{x \in H} c^T x + \max_{x \in M} c^T x\right]$$

$$\text{NOW, } \min_{x \in H} a^T x = \min_{x \in H} \frac{2}{\rho} c^T x = \frac{1}{\rho}\left[\min_{x \in H} c^T x + \underbrace{\min_{x \in H} c^T x}_{\max_{x \in M} c^T x + \rho \ [\text{BY DEF } \rho]}\right]$$

$$= \frac{1}{\rho}\left[\min_{x \in H} c^T x + \max_{x \in M} c^T x + \rho\right]$$

$$= b + 1 \quad [\text{BY DEF OF } b]$$

$$\Rightarrow \min_{x \in H} a^T x = b + 1 \ !$$

Pattern Classification via LP

Proof continued

$$\Rightarrow \text{LET } \rho \doteq \min_{x \in H} c^T x - \max_{x \in M} c^T x > 0$$

$$\text{LET } a \doteq \frac{2}{\rho} c \in \mathbb{R}^n, \quad b \doteq \frac{1}{\rho}\left[\min_{x \in H} c^T x + \max_{x \in M} c^T x\right]$$

Similarly, $\max_{x \in M} a^T x = \max_{x \in M} \frac{2}{\rho} c^T x = \frac{1}{\rho}\Big[\underbrace{\max_{x \in M} c^T x}_{\min_{x \in H} c^T x - \rho} + \max_{x \in M} c^T x\Big]$

$$= \frac{1}{\rho}\Big[-\rho + \underbrace{\min_{x \in H} c^T x + \max_{x \in M} c^T x}_{\rho b}\Big] = -1 + b$$

$$\Rightarrow \max_{x \in M} a^T x = b - 1$$

$$\Rightarrow \exists\, c \in \mathbb{R}^n,\ b \in \mathbb{R} :\ c^T x > b \quad \forall x \in H$$
$$\qquad\qquad\qquad\qquad\qquad \leq b \quad \forall x \in M$$

$$\Rightarrow \min_{x \in H} c^T x > b \geq \max_{x \in M} c^T x$$

$$\Rightarrow \text{LET } p \doteq \min_{x \in H} c^T x - \max_{x \in M} c^T x > 0$$

$$\underset{0.01}{\overset{}{d}} = \min_{x \in H} (c^T x - b) > 0 \qquad a = \frac{1}{d} c$$

$$\overline{\min_{x} c^T x} \qquad b^+ = \frac{1}{d} b$$

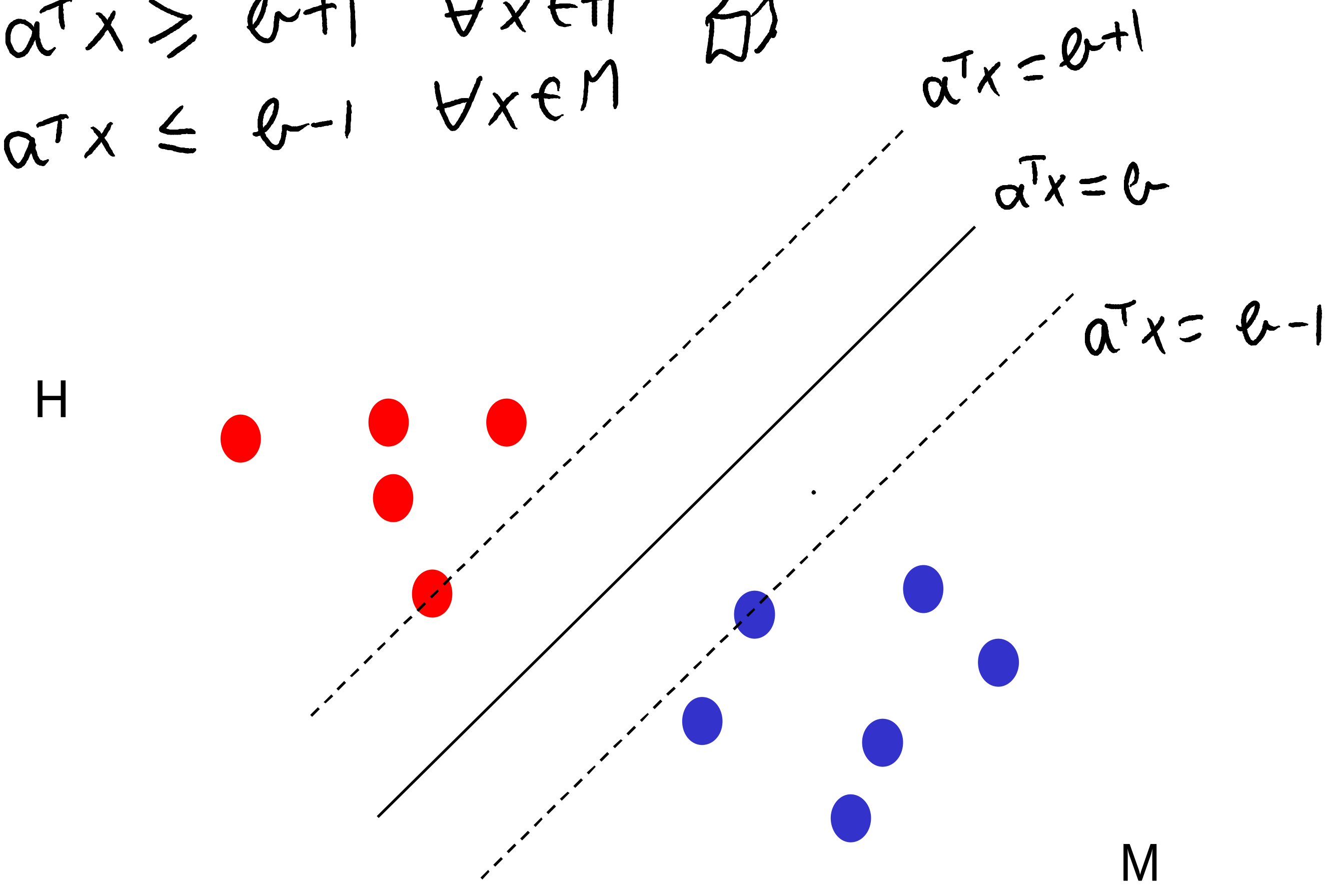
$$e = \max (c^T x - b) \leq 0$$

$$e = -0.2$$

Pattern Classification via LP

Proof continued

$$\Rightarrow \quad a^T x \geq b+1 \quad \forall x \in H$$
$$a^T x \leq b-1 \quad \forall x \in M$$



Pattern Classification via LP

We will see that the following linear problem solves Problem 1 & 2:

GIVEN SETS $H = \{H^1, H^2, \dots, H^h\} \subseteq \mathbb{R}^n$ [Mansgarian 1995]

$M = \{M^1, M^2, \dots, M^m\} \subseteq \mathbb{R}^n$

FIND $y \in \mathbb{R}^h, z \in \mathbb{R}^m, a \in \mathbb{R}^n, b \in \mathbb{R}$ SUCH THAT

$$\min_{\substack{y \in \mathbb{R}^h \\ z \in \mathbb{R}^m \\ a \in \mathbb{R}^n \\ b \in \mathbb{R}}} \frac{1}{h}\left[y_1 + y_2 + \dots + y_h\right] + \frac{1}{m}\left[z_1 + z_2 + \dots + z_m\right]$$

S.T.

$$y_i \geq -a^T H^i + b + 1 \quad \forall i = 1 \dots h$$

$$z_j \geq a^T M^j - b + 1 \quad \forall j = 1 \dots m$$

$$y_i \geq 0 \quad \forall i = 1 \dots h$$

$$z_j \geq 0 \quad \forall j = 1 \dots m$$

Pattern Classification via LP

FIND $y \in \mathbb{R}^k, z \in \mathbb{R}^m, a \in \mathbb{R}^n, b \in \mathbb{R}$ SUCH THAT

$$\min \frac{1}{k}\left[y_1 + y_2 + \dots + y_k\right] + \frac{1}{m}\left[z_1 + z_2 + \dots + z_m\right]$$

$$\text{(LP)} \quad \begin{aligned} & y \in \mathbb{R}^k \quad \text{s.t.} \quad y_i \geq -a^T H^i + b + 1 \quad \forall i = 1 \dots k \\ & z \in \mathbb{R}^m \qquad\quad z_j \geq a^T M^j - b + 1 \quad \forall j = 1 \dots m \\ & a \in \mathbb{R}^n \qquad\quad y_i \geq 0 \quad \forall i = 1 \dots k \\ & b \in \mathbb{R} \qquad\quad\;\; z_j \geq 0 \quad \forall j = 1 \dots m \end{aligned}$$

Theorem 1

H and M are linearly separable iff the optimal value of LP is 0.

Theorem 2

H and M are linearly separable

y^*, z^*, a^*, b^* is an optimal solution of (LP)

$\Rightarrow$ $f(x) = a^{*T}x + b^*$ is a separating hyperplane

Pattern Classification via LP

FIND $y \in \mathbb{R}^k, z \in \mathbb{R}^m, a \in \mathbb{R}^n, b \in \mathbb{R}$ SUCH THAT

$$\left.\begin{array}{lll} \min\limits_{\substack{y \in \mathbb{R}^k \\ z \in \mathbb{R}^m \\ a \in \mathbb{R}^n \\ b \in \mathbb{R}}} & \frac{1}{k}\left[y_1 + y_2 + \ldots + y_k\right] + \frac{1}{m}\left[z_1 + z_2 + \ldots + z_m\right] & \\ \text{s.t.} & y_i \geq -a^T H^i + b + 1 & \forall i = 1 \ldots k \\ & z_j \geq a^T M^j - b + 1 & \forall j = 1 \ldots m \\ & y_i \geq 0 & \forall i = 1 \ldots k \\ & z_j \geq 0 & \forall j = 1 \ldots m \end{array}\right\} (LP)$$

Proof of Theorems 1 and 2

The optimal value of (LP) is 0 $\Longleftrightarrow$

(1) $y^* = 0$

(2) $z^* = 0$

(3) $a^{*T} H^i \geq b^* + 1 \quad \forall i = 1 \ldots k$

(4) $a^{*T} M^j \leq b^* - 1 \quad \forall j = 1 \ldots m$

$\Longleftrightarrow$ H AND M ARE LINEARLY SEPARABLE
$\{x : a^{*T} x - b^* = 0\}$ IS A SEPARATING HYPERPLANE $\square$

Pattern Classification via LP

Application: Breast Cancer Diagnosis

Used at the University of Wisconsin Hospital

[Mangasarian et al 1995]

1. Fluid sample from breast.
2. Placed on a glass and stained the highlight the nuclei of cells
3. Image is taken
4. 30D features: Area, Radius, perimeter, etc

Goal: Classification between benign lumps and malignant lumps

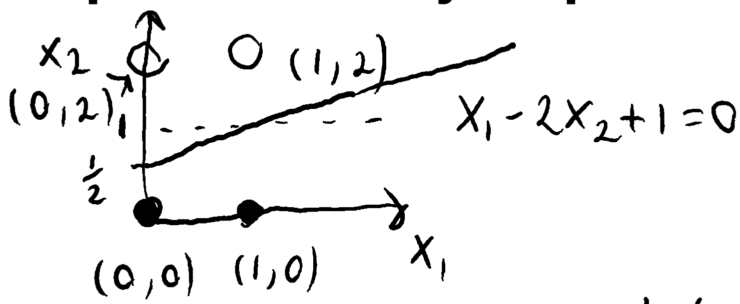
Results: 97.5% accuracy

Pattern Classification via LP

Example 1: Linearly Separable Case

$H = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \quad k = 2$

$M = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \quad m = 2$



LP: MIN $\frac{1}{2}(Y_1 + Y_2) + \frac{1}{2}(Z_1 + Z_2)$ (LP)

a_1, a_2

b

Y_1, Y_2

Z_1, Z_2

S.T. $Y_1 \geq -[a_1, a_2]\begin{bmatrix} H_1^1 \\ H_2^1 \end{bmatrix} + b + 1 = b + 1$ ← $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$Y_2 \geq -[a_1, a_2]\begin{bmatrix} H_1^2 \\ H_2^2 \end{bmatrix} + b + 1 = -a_1 + b + 1$ ← $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Z_1 \geq [a_1, a_2]\begin{bmatrix} M_1^1 \\ M_2^1 \end{bmatrix} - b + 1 = 2a_2 - b + 1$ ← $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$Z_2 \geq [a_1, a_2]\begin{bmatrix} M_1^2 \\ M_2^2 \end{bmatrix} - b + 1 = a_1 + 2a_2 - b + 1$ ← $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Y_1, Y_2, Z_1, Z_2 \geq 0$

AN OPTIMAL SOLUTION

$Y_1 = Y_2 = Z_1 = Z_2 = 0$

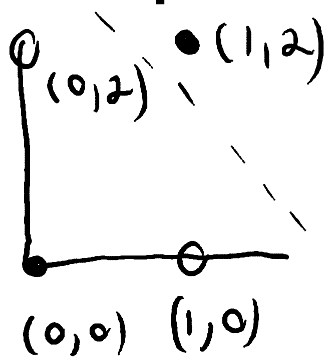
$a^T = [1, -2] \quad b = -1$

$a^T = [0, -1] \quad b = -1$

$0x_1 - 1x_2 + 1 = 0$

Pattern Classification via LP

Example 2: Linearly nonseparable case $H = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ $M = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$



$$\min_{Y_1, Y_2, Z_1, Z_2, a, b} \ \frac{1}{2}(Y_1 + Y_2) + \frac{1}{2}(Z_1 + Z_2)$$

s.t.

$$Y_1 \geq -[a_1 \ a_2] \begin{bmatrix} H_1^1 \\ H_2^1 \end{bmatrix} + b + 1 = b + 1 \qquad \leftarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Y_2 \geq -[a_1 \ a_2] \begin{bmatrix} H_1^2 \\ H_2^2 \end{bmatrix} + b + 1 = -a_1 - 2a_2 + b + 1 \qquad \leftarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Z_1 \geq [a_1 \ a_2] \begin{bmatrix} M_1^1 \\ M_2^1 \end{bmatrix} - b + 1 = 2a_2 - b + 1 \qquad \leftarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$Z_2 \geq [a_1 \ a_2] \begin{bmatrix} M_1^2 \\ M_2^2 \end{bmatrix} - b + 1 = a_1 - b + 1 \qquad \leftarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Y_1, Y_2, Z_1, Z_2 \geq 0$$

AN OPTIMAL SOLUTION:

$Y_1 = 4$, $Y_2 = 0$, $Z_1 = 0$, $Z_2 = 0$

$b = 3$, $a = [2, 1]$ $\Rightarrow$ $2X_1 + X_2 - 3 = 0$ IS THE PROPOSED NONSEPARABLE HYPER PLANE

Linear Programs

- Standard from, Canonical form, Inequality form
- Transforming LPS
    - Pivot transformation

Linear Programs

Inequality form of LPs using matrix notation:

$$\text{MIN [OR MAX]} \quad c^T x \qquad c \in \mathbb{R}^n$$

$$\text{s.t.} \quad Ax \le b \qquad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$$

$$l \le x \le u \qquad l \in \mathbb{R}^n \quad u \in \mathbb{R}^n$$

$$x \in \mathbb{R}^n$$

Standard form of LPs:

$$\text{MIN} \quad c^T x \qquad c \in \mathbb{R}^n$$

$$\text{s.t.} \quad Ax = b$$

$$x \ge 0 \qquad x \in \mathbb{R}^n_{+}$$

$$b \in \mathbb{R}^m_{+}$$

SOMETIMES $b \ge 0$ IS NOT REQUIRED

Theorem: Any LP can be rewritten to an equivalent standard LP

Transforming LPs

Theorem: Any LP can be rewritten to an equivalent standard LP

- Getting rid of inequalities (except variable bounds)

$$x_1 + x_2 \leq 4 \implies x_1 + x_2 + x_3 = 4$$

$$x_3 \geq 0$$

↗ SLACK VARIABLE

- Getting rid of equalities

$$x_1 + 2x_2 = 4 \implies \begin{array}{l} x_1 + 2x_2 \leq 4 \\ x_1 + 2x_2 \geq 4 \end{array}$$

Transforming LPs

- Getting rid of negative variables

$$x \in \mathbb{R} \Rightarrow x = u - v \qquad u \geq 0,\ v \geq 0$$

- Getting rid of bounded variables

$$x \in [2, 5] \Rightarrow \begin{cases} 2 \leq x \\ x \leq 5 \end{cases}$$

- Max to Min

$$\text{MAX } c^T x = -\text{MIN } (-c)^T x$$

- Negative b_i

$$a_i^T x = b_i \Leftrightarrow -a_i^T x = -b_i$$

From Inequality Form to Standard Form

Inequality form

max 2x+3y s.t.

- x + y · 4
- 2x + 5y · 12
- x + 2y · 5
- x, y ¸ 0

if std fm has n vars, m eqns,

then ineq form has

n–m vars and m+(n–m)=n ineqs

(here m = 3, n = 5)

Standard form

max 2x+3y s.t.

- x + y + u = 4
- 2x + 5y + v = 12
- x + 2y + w = 5
- x, y, u, v, w ¸ 0

SLACK VARIABLES

Linear Programing 2

Pivot Transformation

Consider the following problem

$$\text{MIN } Z$$

$$\text{s.t.} \quad 2X_1 + 2X_2 + 2X_3 + X_4 + 4X_5 = Z$$

$$4X_1 + 2X_2 + 13X_3 + 3X_4 + X_5 = 17$$

$$X_1 + X_2 + 5X_3 + X_4 + X_5 = 7$$

$$X_i \geq 0 \quad i = 1, \dots, 5$$

Definition: [Pivot]

- Choose a nonzero element, e.g. $3X_4$
- Use this to eliminate X_4 from the remaining equations
- = Gauss elimination

Pivot Transformation

MIN Z

$$2x_1 + 2x_2 + 2x_3 + x_4 + 4x_5 = Z$$

$$4x_1 + 2x_2 + 13x_3 + \boxed{3x_4} + x_5 = 17 \qquad /3$$

$$x_1 + x_2 + 5x_3 + x_4 + x_5 = 7$$

$$\frac{4}{3}x_1 + \frac{2}{3}x_2 + \frac{13}{3}x_3 + x_4 + \frac{x_5}{3} = \frac{17}{3}$$

$$\left(2 - \frac{4}{3}\right)x_1 + \left(2 - \frac{2}{3}\right)x_2 + \left(2 - \frac{13}{3}\right)x_3 + 0x_4 + \left(4 - \frac{1}{3}\right)x_5 = Z - \frac{17}{3}$$

$$\left(1 - \frac{4}{3}\right)x_1 + \left(1 - \frac{2}{3}\right)x_2 + \left(5 - \frac{13}{3}\right)x_3 + 0x_4 + \left(1 - \frac{1}{3}\right)x_5 = 7 - \frac{17}{3}$$

Pivot Transformation

$$\Rightarrow \frac{2x_1}{3} + \frac{4x_2}{3} - \frac{7x_3}{3} + 0x_4 + \frac{11x_5}{3} = z - \frac{17}{3}$$

$$\frac{4x_1}{3} + \frac{2x_2}{3} + \frac{13x_3}{3} + x_4 + \frac{x_5}{3} = \frac{17}{3}$$

$$-\frac{x_1}{3} + \frac{x_2}{3} + \frac{2x_3}{3} + 0x_4 + \frac{2x_5}{3} = \frac{4}{3} \quad / \times 3 \quad / \times(-2) \quad / \times(-4)$$

After pivot we got an equivalent system: The solution set is the same.

If we pivot again, say in $X_2/3$, then

$$\frac{6}{3}x_1 + 0x_2 - \frac{15}{3}x_3 + 0x_4 + \frac{3}{3}x_5 = z - \frac{33}{3}$$

$$\frac{6}{3}x_1 + 0x_2 + \frac{9}{3}x_3 + x_4 - \frac{3}{3}x_5 = \frac{9}{3}$$

$$-x_1 + x_2 + 2x_3 + 0 + 2x_5 = 4$$

Let us rewrite this:

Canonical Form

$$\left.\begin{aligned}(-Z) + 2x_1 + 0x_2 - 5x_3 + 0x_4 + 1x_5 &= -11\\ 2x_1 + 0x_2 + 3x_3 + 1x_4 - 1x_5 &= 3\\ -x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 &= 4\end{aligned}\right\}(*)$$

Definition [canonical form**]**

- (*) is in canonical form w.r.t (-Z), X_4, X_2 variables
- X_1, X_3, X_5 = Independent (Nonbasic) variables
- -Z, X_4, X_2 = Dependent (Basic) variables.

They are expressed with other variables

Canonical Form

$$
\left.\begin{array}{l}
(-Z) + 2X_1 + 0X_2 - 5X_3 + 0X_4 + 1X_5 = -11 \\
2X_1 + 0X_2 + 3X_3 + 1X_4 - 1X_5 = 3 \\
-X_1 + 1X_2 + 2X_3 + 0X_4 + 2X_5 = 4
\end{array}\right\} (*)
$$

$$X_i \geq 0 \quad i = 1, \ldots, 5$$

- ❑ X_1, X_3, X_5 = Independent (Nonbasic) variables
- ❑ $-Z$, X_4, X_2 = Dependent (Basic) variables.

If we set the nonbasics to zero, then we get values for the basic variables:

$$Z = 11 \qquad X_B = (X_4, X_2) = (3, 4)$$

$$X_N = (X_1, X_3, X_5) = (0, 0, 0)$$

However, if X_1 and X_4 had been chose for pivoting, then

NOT FEASIBLE, SINCE $X_1 < 0$!

$$Z = 3, \quad X_B = (X_1 = -4, X_4 = 11), \quad X_N = (X_2, X_3, X_5) = (0, 0, 0)$$

Canonical Form

Goal of pivots: reduce the original LP problem to canonical form

From canonical form it is easy to find a (basic) solution:

(we just need to set the nonbasic variables to zero)

This basic solution might be

- not feasible (because of the boundary constraint!
  We have to have X_i ¸ 0)
- not optimal (i.e. Z is not minimal)

Pivoting does not alter the solution set.
(After pivots the systems are equivalent)

Canonical Form

$$(-z) + 2x_1 + 0x_2 - 5x_3 + 0x_4 + 1x_5 = -11$$
$$2x_1 + 0x_2 + 3x_3 + 1x_4 - 1x_5 = 3$$
$$-x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 = 4$$

Formal definition of canonical form:

A system of m equations and n variables $(x_j)_{j=1}^{n}$
is in canonical form w.r.t $(x_{j_1}, x_{j_2}, \ldots, x_{j_m})$ VARIABLES

$[-z, x_4, x_2]$
$=$
$[x_{j_1}, x_{j_2}, x_{j_3}]$

$x_4 = x_{j_2}$: 1 COEFF IN EQ 2
0 COEF IN EQ 1, EQ 3

$\Longleftrightarrow$ THE SYSTEM: $I_m x_B + A x_N = b$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -z \\ x_4 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -5 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 4 \end{bmatrix}$$

Canonical Form

Canonical form:

$$I_m X_B + A X_N = b$$

Definition: [Basic solution]

$$X_N = 0$$

Example

$$X_B = (X_1 \dots X_m) \qquad \Rightarrow \qquad x_1 = b_1, \; x_2 = b_2, \; \dots, \; x_m = b_m \qquad x_{m+1} = 0, \; \dots, \; x_n = 0$$

$$X_{NB} = (X_{m+1}, \dots, X_n)$$

THIS BASIC SOLUTION IS FEASIBLE $\Leftrightarrow$ $b_1 \geq 0 \dots b_m \geq 0$

Warming up for the Simplex Algorithm

How to solve LPs if we already have

a canonical form with basic feasible solution?

Simplex Algorithm Phase II

Starting from Canonical Form

Assume that we have a canonical form with feasible basic solution

$$
\begin{array}{llllll}
-z & & & & + c_{m+1}x_{m+1} + \dots + c_j x_j + \dots + c_n x_n & = -z_0 \\
 & x_1 & & & + a_{1,m+1}x_{m+1} + \dots + a_{1j}x_j + \dots + a_{1n}x_n & = b_1 \\
 & & x_2 & & \vdots & \\
 & & & \ddots & & \\
 & & & x_m & + a_{m,m+1}x_{m+1} + \dots + a_{mj}x_j + \dots + a_{mn}x_n & = b_m
\end{array}
$$

$x_i \geq 0$, $1 \leq i \leq n$

c = RELATIVE COST FACTORS [DIFFERENT IN DIFFERENT BASIS]

Using matrix notation:

$$\begin{pmatrix} 1 & 0 & c \\ 0 & I_m & A \end{pmatrix} \begin{pmatrix} -z \\ x_B \\ x_N \end{pmatrix} = \begin{pmatrix} -z_0 \\ b \end{pmatrix}$$

In this canonical form the basic solution is:

$z = z_0$ $\quad x_N = 0$

$x_B = b$

FEASIBLE $\Leftrightarrow$ $b \geq 0$

Improving a Nonoptimal Basic Solution

Let us continue the example

$$(-Z) + 2X_1 + 0X_2 - 5X_3 + 0X_4 + 1X_5 = -11$$
$$2X_1 + 0X_2 + 3X_3 + 1X_4 - 1X_5 = 3 \qquad (*)$$
$$-X_1 + 1X_2 + 2X_3 + 0X_4 + 2X_5 = 4$$

Basic feasible solution: $Z = 11$ $\quad X_B = (X_4, X_2) = (3, 4)$

$X_N = (X_1, X_3, X_5) = (0, 0, 0)$

Goal: min Z, s.t. $X_i \geq 0$

- The relative cost factor of X_3 is (-5)<0
- **Let us see if we can change X_3 from zero to decrease Z**

Improving a Nonoptimal Basic Solution

- The relative cost factor of X_3 is (-5)<0

$$(-Z) + 2X_1 + 0X_2 - 5X_3 + 0X_4 + 1X_5 = -11$$
$$2X_1 + 0X_2 + 3X_3 + 1X_4 - 1X_5 = 3$$
$$-X_1 + 1X_2 + 2X_3 + 0X_4 + 2X_5 = 4$$

$X_B = (X_4, X_2) = (3, 4)$

(*) $X_N = (X_1, X_3, X_5) = (0,0,0)$

$Z = 11$

- **Keep X_3 , and X_B=(-Z,X_2,X_4) as parameters. (X_1,X_5)=(0,0)**

(*) $\Rightarrow$ $Z = 11 - 5X_3$

$X_4 = 3 - 3X_3$

$X_2 = 4 - 2X_3$

X_i ¸ 0, so we can decrease Z by changing those X_i components which have C_i<0 relative cost factors!

We can decrease Z by increasing X_3 from 0,

as long as

$$\begin{cases} X_4 = 3 - 3X_3 \geq 0 \\ X_2 = 4 - 2X_3 \geq 0 \end{cases}$$

Improving a Nonoptimal Basic Solution

$(*) \Rightarrow \quad z = 11 - 5x_3$

$x_4 = 3 - 3x_3$

$x_2 = 4 - 2x_3$

X ¸ 0, so we can decrease Z by changing those X_i components which have $C_i<0$

relative cost factors

We can decrease Z by increasing X_3 from 0,

as long as

$x_4 = 3 - 3x_3 \geq 0$

$x_2 = 4 - 2x_3 \geq 0$

$$
\begin{aligned}
(-z) + 2x_1 + 0x_2 - 5x_3 + 0x_4 + 1x_5 &= -11 \\
2x_1 + 0x_2 + 3x_3 + 1x_4 - 1x_5 &= 3 \\
-x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 &= 4
\end{aligned}
$$

$x_3 = 1$ IS THE BEST IN THIS CASE $\Rightarrow z = 11 - 5 = 6$

$x_4 = 3 - 3 = 0 \Rightarrow x_4$ OUT!

$x_2 = 4 - 2 = 2$

$\frac{b_1}{A_{13}} < \frac{b_2}{A_{23}} \qquad \frac{3}{3} < \frac{4}{2}$

Improving a Nonoptimal Basic Solution

$(*) \Rightarrow$

$$z = 11 - 5x_3$$
$$x_4 = 3 - 3x_3$$
$$x_2 = 4 - 2x_3$$

OLD

$x_B = (x_4, x_2) = (3, 4)$

$x_N = (x_1, x_3, x_5) = (0, 0, 0)$

$z = 11$

NEW

$x_B = (x_3, x_2) = (1, 2)$

$x_N = (x_1, x_4, x_5) = (0, 0, 0)$

$z = 6$

What just happened?

- We brought X_3 into X_B.
- Either X_2 or X_4 can go out into X_N
- We chose X_4 to go out, because that minimizes Z the most
- This is the same as making a pivot on $3X_3$ in (*)

$$\left.\begin{aligned} (-z) + 2x_1 + 0x_2 - 5x_3 + 0x_4 + 1x_5 &= -11 \\ 2x_1 + 0x_2 + \boxed{3x_3} + 1x_4 - 1x_5 &= 3 \\ -x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 &= 4 \end{aligned}\right\} (*)$$

Improving a Nonoptimal Basic Solution

$$\left.\begin{aligned}(-z) + 2x_1 + 0x_2 - 5x_3 + 0x_4 + 1x_5 &= -11\\ 2x_1 + 0x_2 + \boxed{3x_3} + 1x_4 - 1x_5 &= 3\\ -x_1 + 1x_2 + 2x_3 + 0x_4 + 2x_5 &= 4\end{aligned}\right\} \quad (*)\ /:3\ \ /\times\left(-\tfrac{2}{3}\right)\ \ /\times\tfrac{5}{3}$$

$$\left.\begin{aligned}(-z) + \left(\tfrac{10}{3} + 2\right)x_1 + 0x_2 + 0x_3 + \tfrac{5}{3}x_4 + \left(1 - \tfrac{5}{3}\right)x_5 &= -11 + \tfrac{15}{3}\\ \tfrac{2}{3}x_1 + 0x_2 + x_3 + \tfrac{x_4}{3} - \tfrac{x_5}{3} &= 1\\ \left(-\tfrac{4}{3} - 1\right)x_1 + x_2 + 0x_3 - \tfrac{2}{3}x_4 + \left(2 + \tfrac{2}{3}\right)x_5 &= 4 - \tfrac{6}{3}\end{aligned}\right\} \text{ NEW SYSTEM}$$

$\Rightarrow z = 11 - 5 = 6$

$x_B = (x_3, x_2) \qquad x_N = (x_1, x_4, x_5) = (0, 0, 0)$

$\Rightarrow$ now $c_5 = \left(1 - \frac{5}{3}\right) = -\frac{2}{3} < 0 \Rightarrow$ Z CAN BE IMPROVED THE SAME WAY AS BEFORE

Improving a Nonoptimal Basic Solution

$$(-z) + \left(\frac{10}{3} + 2\right)x_1 + 0x_2 + 0x_3 + \frac{5}{3}x_4 + \overbrace{\left(1 - \frac{5}{3}\right)}^{-2/3} x_5 = \underbrace{-11 + \frac{15}{3}}_{6}$$

$$\frac{2}{3}x_1 + 0x_2 + x_3 + \frac{x_4}{3} - \frac{x_5}{3} = 1$$

$$\left(-\frac{4}{3} - 1\right)x_1 + x_2 + 0x_3 - \frac{2}{3}x_4 + \underbrace{\left(2 + \frac{2}{3}\right)}_{8/3} x_5 = \overbrace{4 - \frac{6}{3}}^{2}$$

$(*)$

NOW $c_5 = \left(1 - \frac{5}{3}\right) = -\frac{2}{3} < 0 \Rightarrow z$ CAN BE IMPROVED THE SAME WAY AS BEFORE

- x_5 WILL COME TO $x_B = (x_3, x_2) \Rightarrow x_5, x_2, x_3$ PARAMETERS, $x_1 = x_4 = 0$
- EITHER x_2 OR x_3 WILL GO OUT

$$(*) \Rightarrow z = 6 - \frac{2}{3}x_5$$

$$x_3 = 1 + \frac{1}{3}x_5 \geq 0$$

$$x_2 = 2 - \frac{8}{3}x_5 \geq 0$$

$$\Rightarrow x_5 = \frac{3}{4}$$

$$b_1 - A_{15}x_5 \geq 0$$

$$b_2 - A_{25}x_5 \geq 0$$

Improving a Nonoptimal Basic Solution

$$(-z) + \left(\frac{10}{3}+2\right)x_1 + 0x_2 + 0x_3 + \frac{5}{3}x_4 + \overbrace{\left(1-\frac{5}{3}\right)}^{-2/3}x_5 = \underbrace{-11+\frac{15}{3}}_{5}$$

$$\frac{2}{3}x_1 + 0x_2 + x_3 + \frac{x_4}{3} - \frac{x_5}{3} = 1$$

$$\left(-\frac{4}{3}-1\right)x_1 + x_2 + 0x_3 - \frac{2}{3}x_4 + \underbrace{\left(2+\frac{2}{3}\right)}_{8/3}x_5 = \overbrace{4-\frac{6}{3}}^{2}$$

$\Rightarrow$ now $c_5 = \left(1-\frac{5}{3}\right) = -\frac{2}{3} < 0 \Rightarrow$ Z CAN BE IMPROVED THE SAME WAY AS BEFORE

$c_5 < 0$!

IF $\begin{bmatrix} A_{15} \\ A_{25} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 8/3 \end{bmatrix}$ WERE ALL $\leq 0 \Rightarrow$ Z COULD BE MINIMIZED TO $-\infty$

WE PIVOT ON $\frac{8}{3}x_5 \Rightarrow \begin{cases} x_5 = \frac{3}{4} \text{ TO } X_B \text{ FROM } X_N & \text{IN} \\ x_2 = 0 \text{ TO } X_N \text{ FROM } X_B & \text{OUT} \end{cases}$

Improving a Nonoptimal Basic Solution

$$(-z) + \frac{16}{3}x_1 + 0x_2 + 0x_3 + \frac{5}{3}x_4 - \frac{2}{3}x_5 = -6$$

$$\frac{2}{3}x_1 + 0x_2 + x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = 1$$

$$-\frac{7}{3}x_1 + x_2 + 0x_3 - \frac{2}{3}x_4 + \boxed{\frac{8}{3}x_5} = 2$$

$/\times\frac{3}{8}$ $\quad /\times\frac{1}{8}$ (+) $\quad /\times\frac{2}{8}$ (+)

$$\Rightarrow (-z) + \left(\frac{16}{3} - \frac{7}{12}\right)x_1 + \frac{x_2}{4} + 0x_3 + \left(\frac{5}{3} - \frac{2}{12}\right)x_4 + 0x_5 = -6 + \frac{2}{4}$$

$$\left(\frac{2}{3} - \frac{7}{24}\right)x_1 + \frac{x_2}{8} + x_3 + \left(\frac{1}{3} - \frac{2}{24}\right)x_4 + 0x_5 = 1 + \frac{2}{8}$$

$$-\frac{7}{8}x_1 + \frac{3}{8}x_2 + 0x_3 - \frac{2}{8}x_4 + x_5 = \frac{6}{8}$$

$$\Rightarrow z = 6 - \frac{2}{4} = \frac{11}{2} \qquad x_B = (x_3, x_5)$$

$$x_3 = 1 + \frac{2}{8} = \frac{5}{4} \qquad x_N = (x_1, x_2, x_4) = 0$$

$$x_5 = \frac{6}{8} = \frac{3}{4}$$

The Simplex Algorithm (Phase II)

Key components of the simplex algorithm

1. Optimality test

   IF $C_j \geq 0 \ \forall j \Rightarrow$ BASIC FEASIBLE SOLUTION IS OPTIMAL

2. In each step one variable in, one variable out
   (Traveling on the neighboring corners of the polytope)

   BRING ONE VARIABLE TO x_B FROM x_N
   — " — x_N — " — x_B

3. The adjusted values have to be nonnegative

The Simplex Algorithm (Phase 2)

Assume that we start from a **feasible canonical form**:

$$(-z) + 0x_B + c^T x_N = -z_0$$

$$I x_B + \Delta x_N = b$$

The initial feasible solution is: $x_B = b \geq 0$, $x_N = 0$, $z = z_0$

Steps of the Simplex algorithm

(1) Smallest reduced cost

FIND $s = \arg\min_j c_j$

$$c_s = \min_j c_j$$

The Simplex Algorithm

Steps of the Simplex algorithm

(2) Test for optimality

IF $C_S \geq 0 \Rightarrow$ REPORT THE BASIC FEASIBLE SOLUTION AS OPTIMAL AND STOP

(3) Incoming variable

IF $C_S < 0 \Rightarrow S$ IS THE INDEX OF THE INCOMING VARIABLE

/BTW ANY $j: c_j < 0$ COULD DEFINE AN INCOMING VARIABLE/

(4) Test for unbounded Z

WE WILL USE THE STH COLUMNS OF A

IF $A_{\cdot S} \leq 0 \Rightarrow z^* = -\infty$ [THE OPTIMAL SOLUTION IS UNBOUNDED]

$X_S \rightarrow \infty \Rightarrow z \rightarrow -\infty$

The Simplex Algorithm

Steps of the Simplex algorithm

(5) Outgoing variable

- This r will show the outgoing variable
- The basic variable in the r^th row of A

LET $r = \arg\min_{\{i \mid A_{is} > 0\}} \frac{b_i}{A_{is}}$ ≥ 0

WE PIVOT IN POSITIVE A_{rs}!

REMEMBER, IF $A_{\cdot s} \leq 0 \Rightarrow z^* = -\infty$

Lemma [New basic solution remains feasible**]**

THIS WAY, FOR THE NEW b_j $(= \tilde{b}_j)$ WE HAVE $\tilde{b}_j \geq 0$

Proof $\tilde{b}_j = b_j + b_r \frac{-A_{js}}{A_{rs}} \geq 0$ SINCE IF $A_{js} \leq 0$ ✓

IF $A_{js} > 0 \Rightarrow \frac{b_j}{A_{js}} \geq \frac{b_r}{A_{rs}}$

The Simplex Algorithm

Steps of the Simplex algorithm

(6) Pivot on A_{rs}

- This gives us new basic feasible solution
- We do this pivot regardless if Z changes or not

$$(-\tilde{z}) = (-z_0) + b_r \cdot \frac{(-c_s)}{A_{rs}}$$

IF $b_r = 0 \Rightarrow$ NO CHANGE IN z!

The Simplex Algorithm

Steps of the Simplex algorithm

- If zero change in the objective Z, then cycling can happen
- Bland rule can avoid cycling

Bland's rule: Whenever the pivot in the simplex method would result in a zero change of the objective Z, do the following:

(i) Incoming column:

CHOOSE PIVOT COLUMN J=S WITH $\bar{C}_j < 0$ HAVING THE SMALLEST INDEX j

(ii) Outgoing column:

CHOOSE THE ELIGIBLE COLUMN THAT HAS THE SMALLEST INDEX

The Simplex Algorithm Summary

Theorem:

A basic feasible solution is optimal with total cost Z_0,

if all relative cost factors (C_j, j=1,...,n) are nonnegative.

Proof:

$$
\begin{array}{llllll}
-z & & & & + c_{m+1}x_{m+1} + \dots + c_j x_j + \dots + c_n x_n & = -z_0 \\
 & x_1 & & & + a_{1,m+1}x_{m+1} + \dots + a_{1j}x_j + \dots + a_{1n}x_n & = b_1 \\
 & & x_2 & & \vdots & \\
 & & & \ddots & & \\
 & & & x_m & + a_{m,m+1}x_{m+1} + \dots + a_{mj}x_j + \dots + a_{mn}x_n & = b_m
\end{array}
$$

$x_i \geq 0$

$1 \leq i \leq n$

Theorem:

A basic feasible solution is the **unique** optimal solution with total cost Z_0,

if $C_j > 0$ for all nonbasic variables.

The Simplex Algorithm Summary

Theorem:

Assuming "non-degeneracy" at each iteration (b_j>0, j=1,...,m),

the simplex algorithm will converge in finite steps.

Proof:

There are only finite many basis, and because of "non-degeneracy",

cycling cannot happen.

Remark:

- If we use infinite-precision arithmetic, then we can find the exact solution. (No approximation used)
- Interior point methods can only converge to an epsilon ball that contains the solution.

The full Simplex Algorithm

So far we have assumed that a basic feasible solution in canonical form is available to start the algorithm...

The Simplex Algorithm Summary

The simplex method can be applied to a Linear Program in standard form:

$\min c^T x$

$x \geq 0$

$s.t.\ Ax = b$

Phase I:

- Find a starting basic feasible solution in canonical form

  and detect redundancies $3x_1 + 4x_2 = 5$, $6x_1 + 8x_2 = 10$

- or determine if such solution doesn't exist

  detect inconsistencies $3x_1 + 4x_2 = 5$, $6x_1 + 8x_2 = 7$

Phase II:

If starting basic solution found, then

- find an optimal solution
- or show that $Z \rightarrow -1$ is possible

WE HAVE DISCUSED THIS

The Simplex Algorithm Phase I

Example

$$2x_1 + 1x_2 + 2x_3 + x_4 + 4x_5 = z \rightarrow \text{MIN}, \quad x \geq 0 \quad (*1)$$

$$\text{s.T} \quad 4x_1 + 2x_2 + 13x_3 + 3x_4 + x_5 = 17$$

$$x_1 + x_2 + 5x_3 + x_4 + x_5 = 7$$

Goal: We want to find a feasible solution

Phase I:

(i) Forget the cost function $c^T x$.

(ii) Introduce $X_6, X_7 \geq 0$. [One variable for each row]

(iii) Solve

$$\text{MIN} \quad x_6 + x_7 = w$$

$$x_i \geq 0 \quad i = 1..7$$

$$\text{s.T} \quad 4x_1 + 2x_2 + 13x_3 + 3x_4 + x_5 + x_6 + 0x_7 = 17$$

$$x_1 + x_2 + 5x_3 + x_4 + x_5 + 0x_6 + x_7 = 7$$

The Simplex Algorithm Phase I

$$\left.\begin{array}{ll} & \text{MIN } x_6 + x_7 = w \\ & x_i \geq 0 \quad i = 1..7 \\ \text{s.t} & 4x_1 + 2x_2 + 13x_3 + 3x_4 + x_5 + x_6 + 0x_7 = 17 \\ & x_1 + x_2 + 5x_3 + x_4 + x_5 + 0x_6 + x_7 = 7 \end{array}\right\} (*2)$$

Theorem: (*2) has feasible optimal solution such that $X_6=X_7=0$

iff (*1) has feasible solution

Remarks:

- (*2) is easy to convert to a feasible canonical solution (We will see)
- We can find its optimal solution ($X_6=X_7=0$) with the Phase II algorithm
  This is a feasible solution of (*1)

Tableaux

$$\text{MIN } x_6 + x_7 = w$$

$$x_i \geq 0 \quad i = 1..7$$

s.T

$$4x_1 + 2x_2 + 13x_3 + 3x_4 + x_5 + x_6 + 0x_7 = 17$$

$$x_1 + x_2 + 5x_3 + x_4 + x_5 + 0x_6 + x_7 = 7$$

$(\times 2)$

Basic variable	Objective (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
	1	0	0	0	0	0	1	1	0
	0	4	2	13	3	1	1	0	17
	0	1	1	5	1	1	0	1	7

It's easy to convert this to a feasible canonical form:

Basic variable	Objective (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

Tableaux

Now, we can run the Phase 2 algorithm on this table to get a feasible solution of (*1):

B. var	Obj. (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

-18 IS THE SMALLEST $C_j < 0$ ⇒ PIVOT IN COLUMN 3

$\frac{17}{13} < \frac{7}{5}$ ⇒ PIVOT IN 13; X_3 IN; X_6 OUT

Tableaux

Now, we can run the Phase 2 algorithm on this table to get a feasible solution of (*1):

B. var	Obj. (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

⇓

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5+18/13*4	-3+18/13*2	0	-4+18/13*3	-2+18/13	18/13	0	-24+18/13*7
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-5/13*4	1-5/13*2	0	1-5/13*3	1-5/13	-5/13	1	7-5/13*7

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5+18/13*4	-3+18/13*2	0	-4+18/13*3	-2+18/13	18/13	0	-24+18/13*7
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-5/13*4	1-5/13*2	0	1-5/13*3	1-5/13	-5/13	1	7-5/13*7

Let us simplify this Table a little:

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-7/13	-3/13	0	2/13	-8/13	18/13	0	-6/13
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-7/13	3/13	0	-2/13	8/13	-5/13	1	6/13

↑ PIVOT

-8/13 IS THE SMALLEST AMONG ALL $c_j < 0 \Rightarrow X_5$ IN

$$\frac{6/13}{8/13} < \frac{17/13}{1/13} \Rightarrow X_7 \text{ OUT}$$

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	0	0	0	0	0	1	1	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

All the relative costs are nonnegative) optimal feasible solution.

Phase I is finished.

$-w + x_6 + x_7 = 0$

$x_3 = 5/4$

$x_5 = 3/4$

$x_1, x_2, x_4, x_6, x_7 = 0$

FEASIBLE BASIC SOLUTION OF THE ORIGINAL (*1) PROBLEM

⇓

PHASE II CAN BE STARTED

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	0	0	0	0	0	1	1	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

ORIGINAL COST FUNCTION: $z = 2x_1 + 1x_2 + 2x_3 + x_4 + 4x_5 \rightarrow \min$, $x \geq 0$

B. var	Obj (-z)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-z	1	2	1	2	1	4	0	0	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

Tableaux

B. var	Obj (-z)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-z	1	2	1	2	1	4	0	0	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

-2x
-4x

Make it to canonical form and continue with Phase II...

Do not make pivots in the column of X_6 and X_7

Simplex Algorithm with Matlab

```
f = [-5  -4  -6]';

A =  [ 1 -1  1
       3  2  4
       3  2  0 ];

b = [20  42 30]';

lb = zeros(3,1);

options = optimset('LargeScale','off','Simplex','on');

[x,fval,exitflag,output,lambda] = linprog(f,A,b,[],[],lb,[],[],options);
```

Relevant Books

- Luenberger, David G. *Linear and Nonlinear Programming*. 2nd ed. Reading, MA: Addison Wesley, 1984. ISBN: 0201157942.
- Bertsimas, Dimitris, and John Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific Press, 1997. ISBN: 1886529191.
- Dantzig, Thapa: Linear Programming

Summary

- Linear programs:
  - standard form,
  - canonical form
- Solutions:
  - Basic, Feasible, Optimal, Degenerate
- Simplex algorithm:
  - Phase I
  - Phase II
- Applications:
  - Pattern classification