Administrivia

Simplex Algorithm in 1 Slide

Canonical form:

If we do pivot in $A_{r,s} > 0$, where $c_s < 0$

New cost value:

New b vector:

The full Simplex Algorithm

So far we have assumed that a basic feasible solution in canonical form is available to start the algorithm…

The Simplex Algorithm Phase I

Example

Goal: We want to find a feasible solution

Phase I:

(i) Forget the cost function $c^T x$.

(ii) Introduce $X_6, X_7 \geq 0$. [One variable for each row]

(iii) Solve

The Simplex Algorithm Phase I

Theorem: (*2) has feasible optimal solution such that $X_6=X_7=0$

iff (*1) has feasible solution

Remarks:

- (*2) is easy to convert to a feasible canonical solution (We will see)
- We can find its optimal solution ($X_6=X_7=0$) with the Phase II algorithm
  This is a feasible solution of (*1)

Tableaux

Basic variable	Objective (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
	1	0	0	0	0	0	1	1	0
	0	4	2	13	3	1	1	0	17
	0	1	1	5	1	1	0	1	7

It's easy to convert this to a feasible canonical form:

Basic variable	Objective (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

Tableaux

Now, we can run the Phase 2 algorithm on this table to get a feasible solution of (*1):

B. var	**Obj. (-w)**	X_1	X_2	X_3	X_4	X_5	X_6	X_7	**RHS**
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

Tableaux

Now, we can run the Phase 2 algorithm on this table to get a feasible solution of (*1):

B. var	**Obj. (-w)**	X_1	X_2	X_3	X_4	X_5	X_6	X_7	**RHS**
-w	1	-5	-3	-18	-4	-2	0	0	-24
X_6	0	4	2	13	3	1	1	0	17
X_7	0	1	1	5	1	1	0	1	7

B. var	**Obj (-w)**	X_1	X_2	X_3	X_4	X_5	X_6	X_7	**RHS**
-w	1	-5+18/13*4	-3+18/13*2	0	-4+18/13*3	-2+18/13	18/13	0	-24+18/13*7
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-5/13*4	1-5/13*2	0	1-5/13*3	1-5/13	-5/13	1	7-5/13*7

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-5+18/13*4	-3+18/13*2	0	-4+18/13*3	-2+18/13	18/13	0	-24+18/13*7
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-5/13*4	1-5/13*2	0	1-5/13*3	1-5/13	-5/13	1	7-5/13*7

Let us simplify this Table a little:

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	-7/13	-3/13	0	2/13	-8/13	18/13	0	-6/13
X_3	0	4/13	2/13	1	3/13	1/13	1/13	0	17/13
X_7	0	1-7/13	3/13	0	-2/13	8/13	-5/13	1	6/13

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	0	0	0	0	0	1	1	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

All the relative costs are nonnegative $\Rightarrow$ optimal feasible solution.

Phase I is finished.

Tableaux

B. var	Obj (-w)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-w	1	0	0	0	0	0	1	1	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

B. var	Obj (-z)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-z	1	2	1	2	1	4	0	0	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

Tableaux

B. var	Obj (-z)	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
-z	1	2	1	2	1	4	0	0	0
X_3	0	3/8	1/8	1	1/4	0	-1/8	0	5/4
X_5	0	-7/8	3/8	0	-1/4	1	-5/8	1/8	3/4

Make it to canonical form and continue with Phase II…

Do not make pivots in the column of X_6 and X_7

Simplex Algorithm with Matlab

```
f = [-5  -4  -6]';

A =  [ 1 -1  1
        3  2  4
        3  2  0 ];

b = [20  42 30]';

lb = zeros(3,1);

options = optimset('LargeScale','off','Simplex','on');

[x,fval,exitflag,output,lambda] = linprog(f,A,b,[],[],lb,[],[],options);
```

Relevant Books

- Luenberger, David G. *Linear and Nonlinear Programming*. 2nd ed. Reading, MA: Addison Wesley, 1984. ISBN: 0201157942.
- Bertsimas, Dimitris, and John Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific Press, 1997. ISBN: 1886529191.
- Dantzig, Thapa: Linear Programming

Summary

- Linear programs:
  - standard form,
  - canonical form
- Solutions:
  - Basic, Feasible, Optimal, Degenerate
- Simplex algorithm:
  - Phase I
  - Phase II
- Applications:
  - Pattern classification

Convex Optimization
CMU-10725

4. Convexity Part I

Barnabás Póczos & Ryan Tibshirani

Goal of this lecture

- **Review of Convex sets & Convex functions**
  - Definition
  - Examples
  - Basic properties

Books to Read:

- **Boyd and Vandenberghe**: Convex Optimization, Chapters 2 & 3
- **Rockafellar**: Convex Analysis

Line and Line Segments

Definition [Line]:

$$\{x \in \mathbb{R}^n | x = \theta x_1 + (1-\theta) x_2, \theta \in \mathbb{R}\}$$

Definition [Line segment]:

$$\{x \in \mathbb{R}^n | x = \theta x_1 + (1-\theta) x_2, \theta \in [0,1]\}$$

Affine Sets

Definition [Affine set]:

A set C is affine if for any $x_1, x_2 \in C$ the line through x_1 and x_2 is in C, i.e. $\theta x_1 + (1-\theta)x_2 \in C$, $(\theta \in \mathbb{R})$

Definition [Affine hull of set C]:

Theorem [Affine hull]:

The Aff[C] is the smallest affine set that contains C

Affine Sets Example

Example [Solutions of linear equations]:

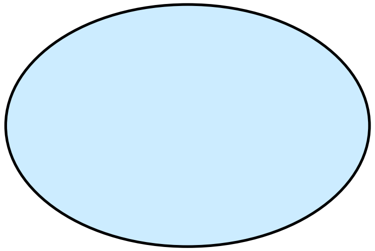
The solution set of a system of linear equations is an affine set

Solution set:

Proof:

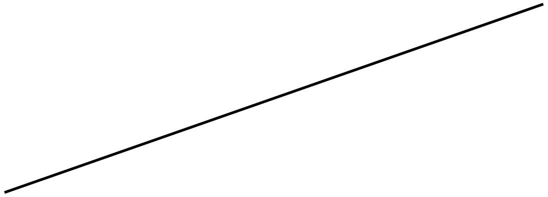
Boundaries

Definition [x on boundary of C (∂C)]:



Definition [x in interior of C]:

Definition [*relative* interior (rel int C)]:



Boundaries

Definition [closure of C (cl C)]:

Definition [*relative* boundary of C (rel ∂C)]:

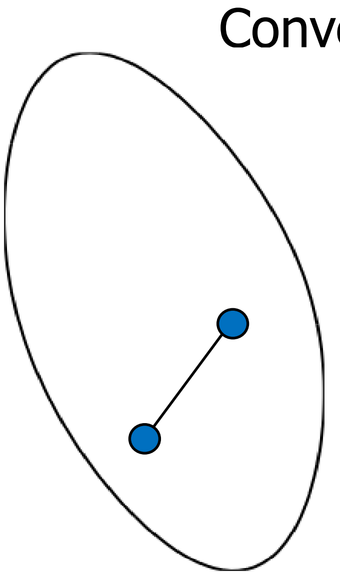
Open and Closed Sets

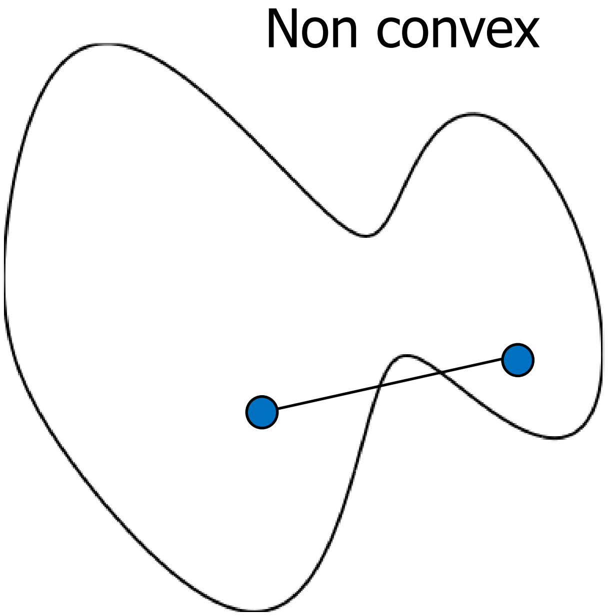
Definition [C closed]:

Definition [C open]:

Definition [C compact]:

Convex sets

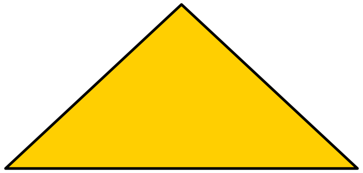




Definition [Convex set]:

Definition [Strictly convex set]:

Example [Convex, but not strictly convex]:



Examples of Convex Sets

- empty set:
- singleton set:
- complete space:

- lines:
- line segments:
- hyperplanes:
- halfspaces:

Examples of Convex Sets

- Euclidian balls:

- L_p balls, $p \geq 1$

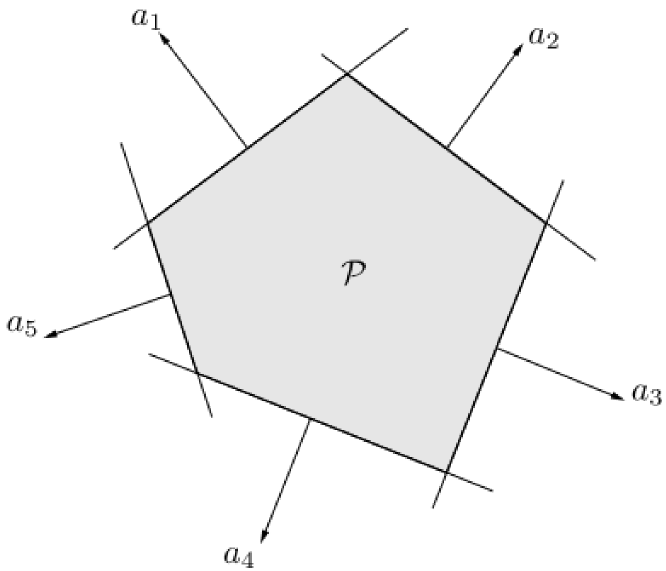
- L_p balls $0<p<1$

Examples of Convex Sets

- Polyhedron: the solution set of a finite number of linear equalities and inequalities

Matrix notation:

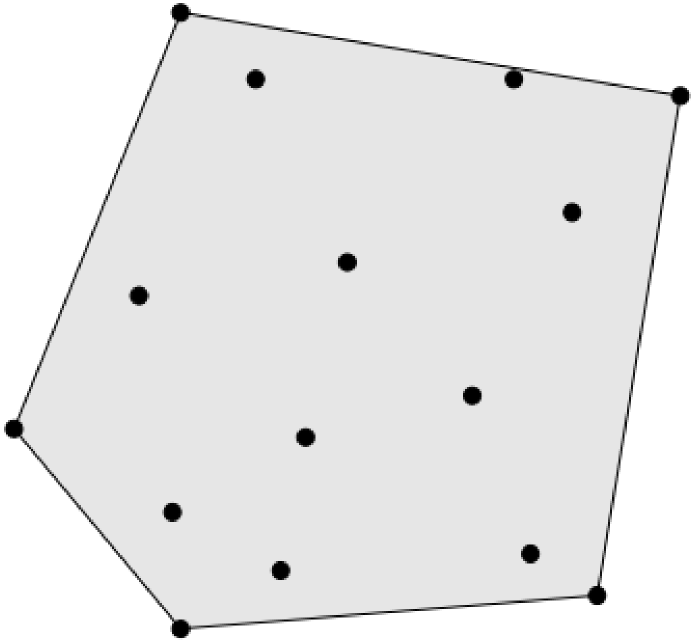
- Polytope: bounded polyhedron

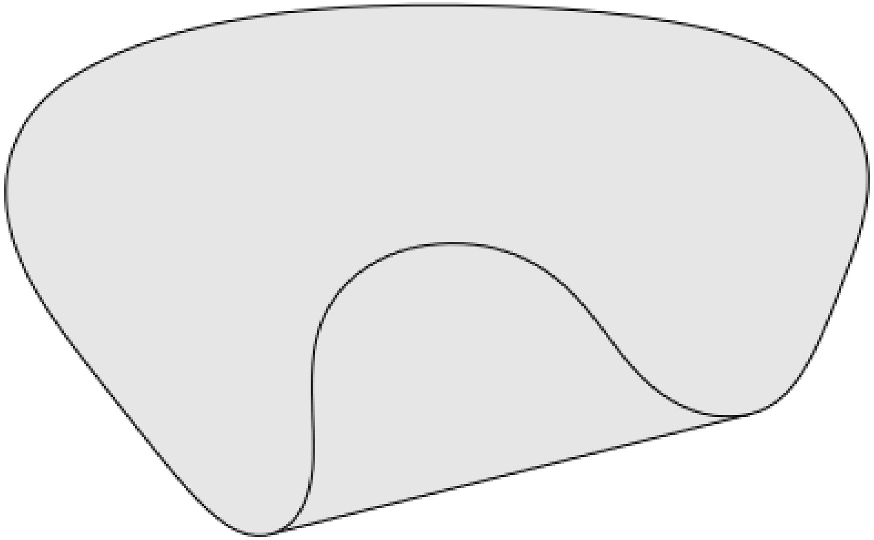


Intersection of halfspaces

& hyperplanes

Convex hull





Definition [Convex hull]:

Convex hull

Convex hull properties:

Examples of Convex Sets

- Simplex:

Convex Combination

Infinite many sums

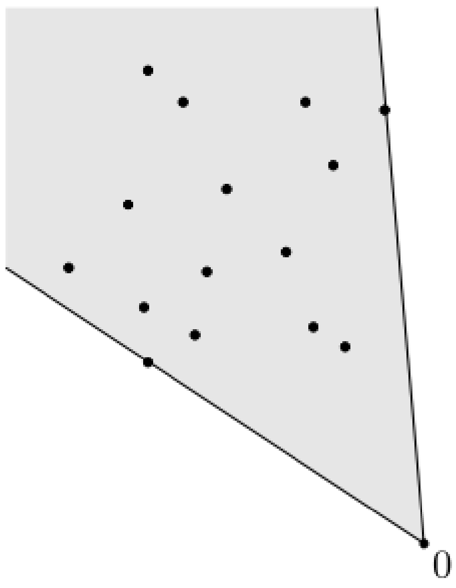
Integrals (Expected value)

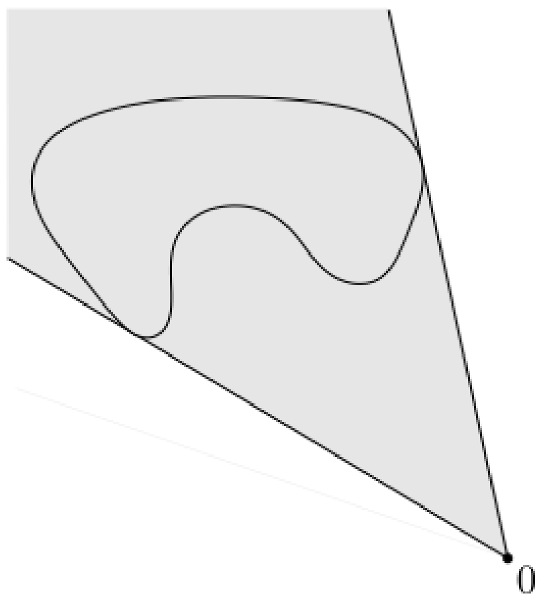
Cones

Definition [Cone]:

Definition [Convex Cone]:

Definition [Conic hull]:





Example: PSD matrices

Definition [Positive semi definite matrix]:

Theorem [eigenvalues]:

A symmetric matrix A is positive definite iff all its eigenvalues are positive

Partial ordering of square matrices**:**

For arbitrary square matrices M, N we write $M \geq N$

if $M - N \geq 0$; i.e., $M - N$ is positive semi-definite.

Example: PSD matrices

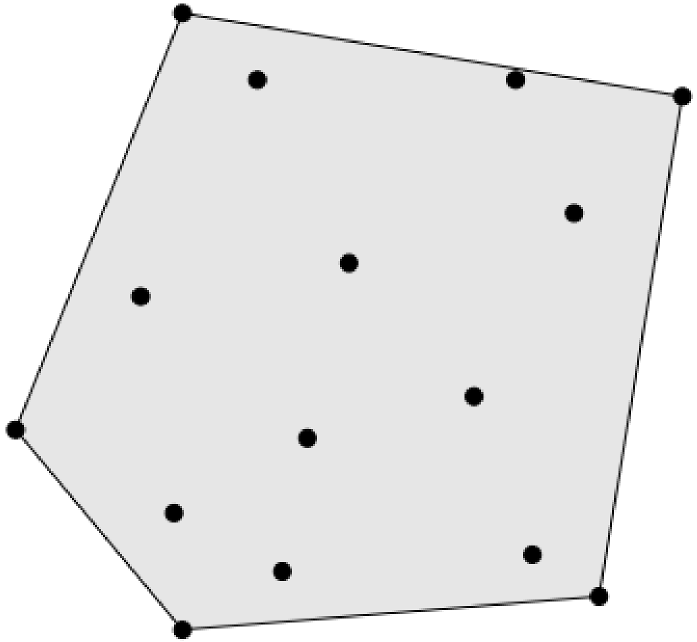
Theorem [Cone of PSD matrices]:

The set of symmetric, PSD matrices form a convex cone:

Proof:

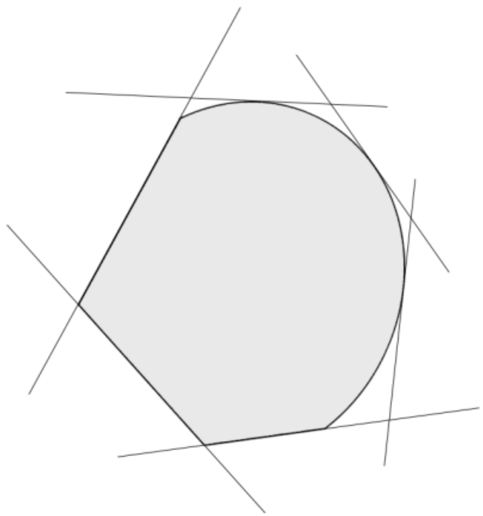
Convex set representation with convex hull

Theorem: [Representation of a closed convex set with a convex hull]



Dual representation

Theorem: [Representation of a closed convex set with half spaces]



Convexity-preserving set operations

- Translation
- Scaling
- Intersection

Convexity-preserving set operations

- Affine function
  - E.g. projection, dropping coordinates
- Set sum
- Direct sum

Convex Optimization
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5. Convexity Part II

Barnabás Póczos & Ryan Tibshirani

Conevex Hull

Definition 1

Note:

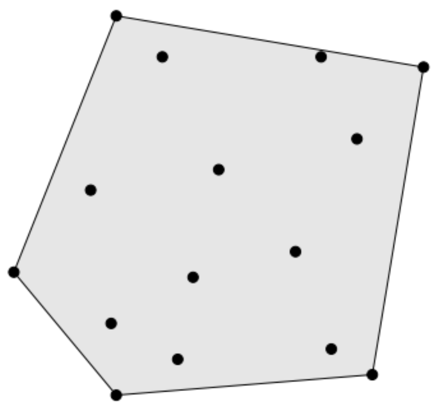
- If C is a finite set, then this is closed polyhedron.
- If C contains infinite many points, then this can be open, closed, or none of them

Theorem [Definition 2, Primal representation]

A closed convex set is the intersection of all the closed half spaces containing S

Convex set representation with convex hull

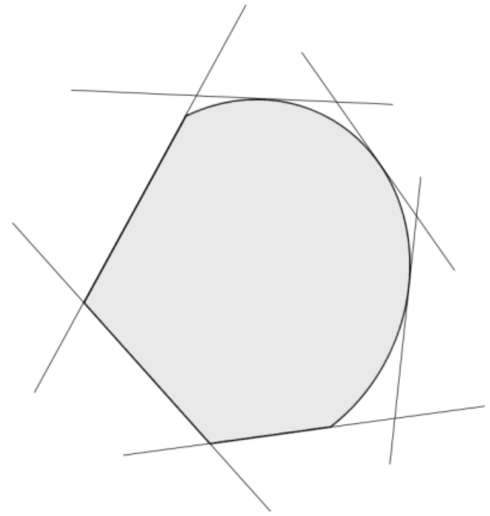
Theorem: [Representation of a closed convex set with a convex hull]



Convex hull = convex combination of possibly infinite many points in the set.

Dual representation

Theorem: [Representation of a closed convex set with half spaces]



A closed convex set is the intersection of all the closed half spaces containing S

Convexity-preserving set operations

- Perspective projection (pinhole camera)

Convexity-preserving set operations

- Linear-fractional function
  (perspective function with affine function)

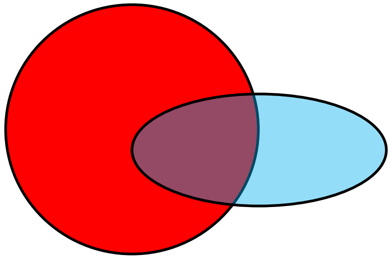
Theorem: [Image of Linear fractional function]

Convexity-preserving set operations

Application: [Conditional probabilities]

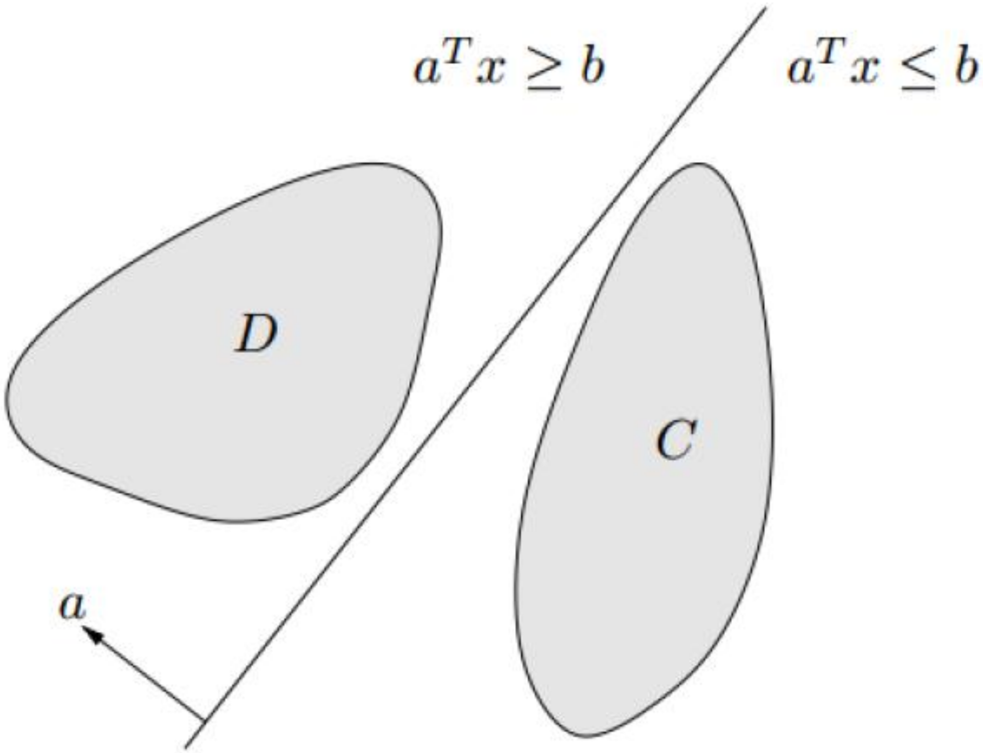
Convexity-preserving set operations

- Union doesn't preserve convexity



Separating hyperplane thm

Theorem: [Separating hyperplane theorem]



Separating hyperplane thm

Definition: [Strong separation]

Definition: [Proper separation],

Definition: [Strict separation]

It "strictly separates" them if neither one touches the hyperplane.

Separating hyperplane thm

Theorem: [Strong separation theorem]

Counterexample:

Why do we need at least one bounded set?

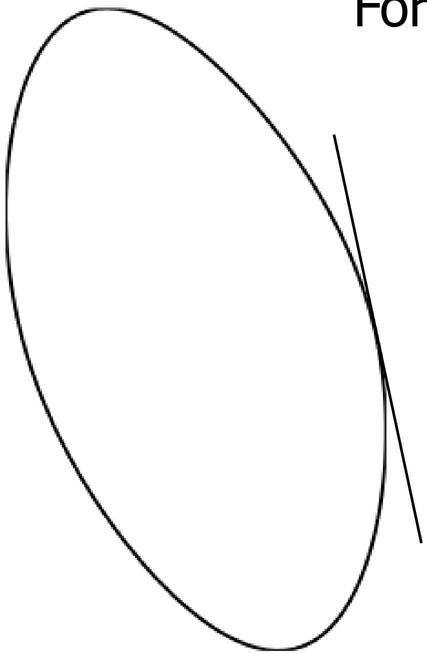
Separating hyperplane thm II

Theorem: [Strong separation theorem II]

Supporting hyperplane thm

Theorem: [Supporting hyperplane theorem]

For any point x_0 on the boundary of convex C



Theorem: [Partial converse of the supporting hyperplane theorem]

Proving a set convex

- Use definition directly

- Represent as convex hull

- Represent as the intersection of halfspaces

- Supporting hyperplane partial converse:
  - C closed, nonempty interior, has supporting hyperplane at all boundary points $\Rightarrow$ C convex

- Build C up from simpler sets using convexity-preserving operations

Convex functions

Convex functions

Definition [convex function]:

Definition [strictly convex function]:

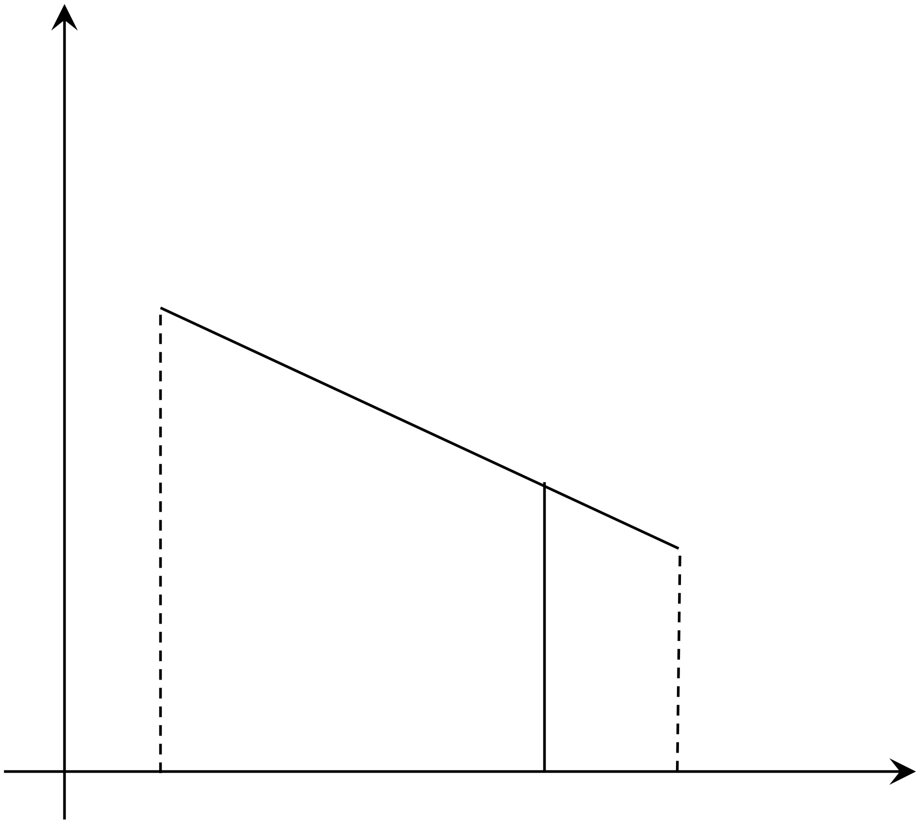
Concave functions

Definition [concave function]:

-f is convex

Convex functions

Geometric interpretation



Strongly convexity

Definition:[m-strongly convex function (m>0)

$$(\nabla f(x) - \nabla f(y))^T(x-y) \geq m\|x-y\|_2^2$$

An equivalent condition:

$$f(y) \geq f(x) + \nabla f(x)^T(y-x) + \frac{m}{2}\|y-x\|_2^2$$

Without gradient: $\forall t \in [0,1]$

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}mt(1-t)\|x-y\|_2^2$$

With Hessian:

$\nabla^2 f(x) \succeq mI$ for all x in the domain

A strongly convex function is also strictly convex, but not vice-versa.

Examples

$f(x) = x^4$: convex, strictly convex, not strongly convex.

$f(x) = |x|$: convex, not strictly convex.

Examples: Convex functions

Convex

Concave

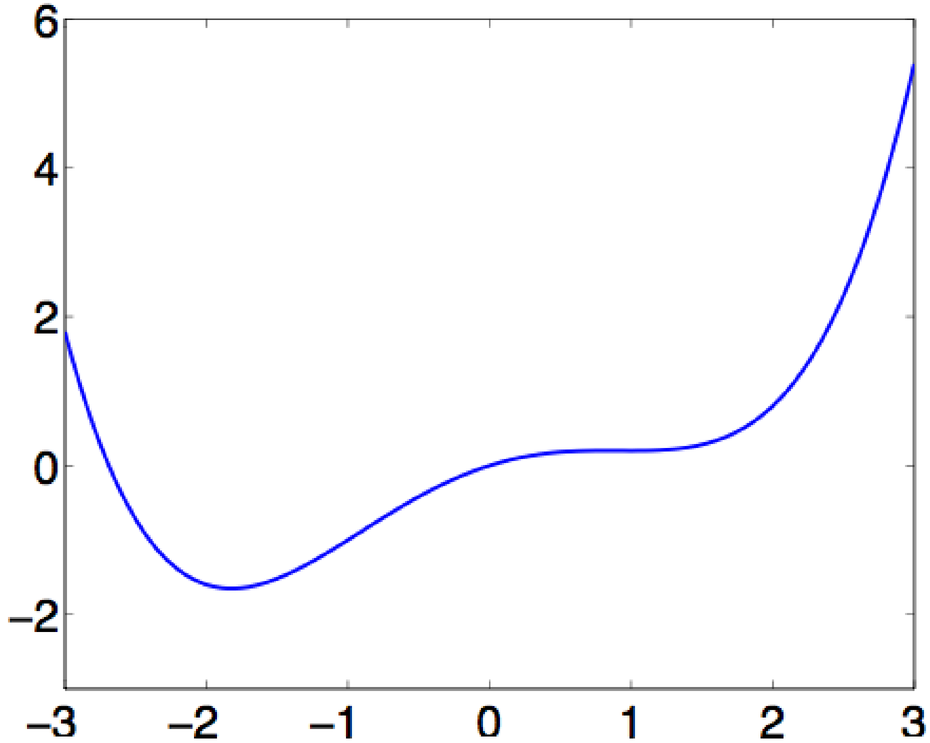
Extended reals

We can extend f from dom f to R^n without changing its convexity

Theorem:

Epigraph

Definition [epigraph]:

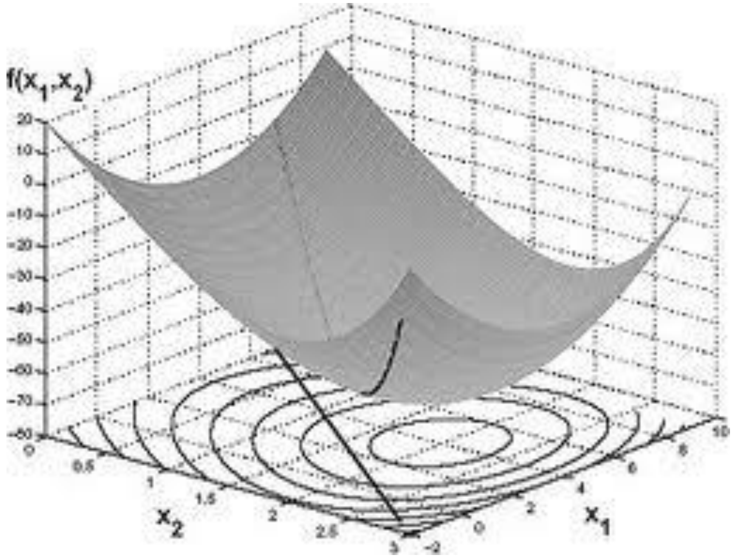


Theorem [convexity of the epigraph]:

Convex Function Properties

0th order characterization

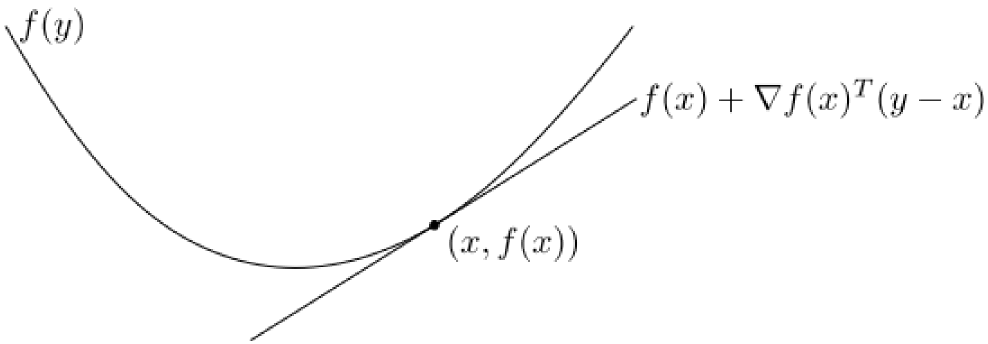
This is useful, because we only need to check the convexity of 1D functions.



Graph courtesy of Prof. Robert Freund

Convex Function Properties

1st order characterization



The 1st oder Taylor approximation is a global underestimator of f.

Corollary:

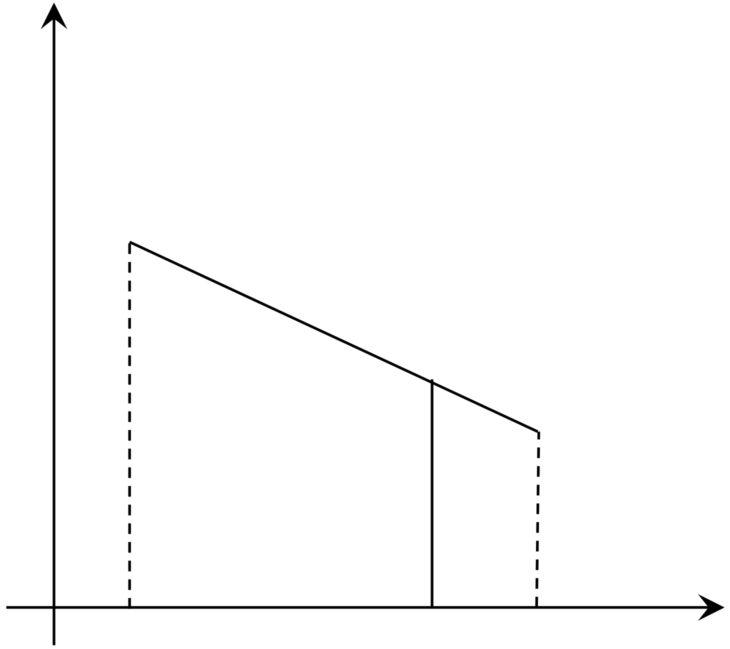
Convex Function Properties

2nd order characterization

Lemma

Jensen's inequality

Theorem



Proving a function convex

- Use definition directly
- Prove that epigraph is convex via set methods
- 0^{th}, 1^{st}, 2^{nd} order convexity properties
- Construct f from simpler convex fns using convexity-preserving ops

Convexity-preserving fn ops

Nonnegative weighted sum

If f_1, f_2 cvx, $w_i \geq 0 \Rightarrow h(x) = w_1 f_1(x) + w_2 f_2(x)$ cvx

Pointwise max/sup

If f, g cvx, $\Rightarrow m(x) = \max\{f(x), g(x)\}$ cvx

Extension of pointwise max/sup

If $f(x, y)$ is convex in x for each y

$\Rightarrow g(x) = \sup_{y \in C} f(x, y)$ is convex in x,

provided $g(x) > -\infty$ for some x.

Convexity-preserving fn ops

Affine map

If $f : \mathbb{R}^n \to \mathbb{R}$ is cvx, $\Rightarrow g(x) = f(Ax + b)$ is cvx,
where $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$.

Composition

If f, g are cvx, and g is non-decreasing,
$\Rightarrow h(x) = g(f(x))$ is cvx.

If f is concave and g is cvx and non-increasing,
$\Rightarrow h(x) = g(f(x))$ is cvx.

Perspective map

If $f(x)$ is convex, $\Rightarrow g(x,t) = tf(x/t)$ is convex.

Summary

- **Convex sets**
  - Representation:
    - convex hull, intersect hyperplanes
  - supporting, separating hyperplanes
  - operations that preserve convexity

- **Convex functions**
  - epigraph
  - 0 orders, 1st order, 2nd order conditions
  - operations that preserve convexity