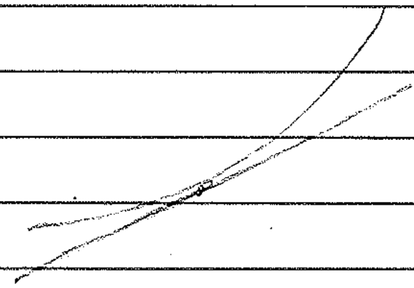


C $\quad I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$

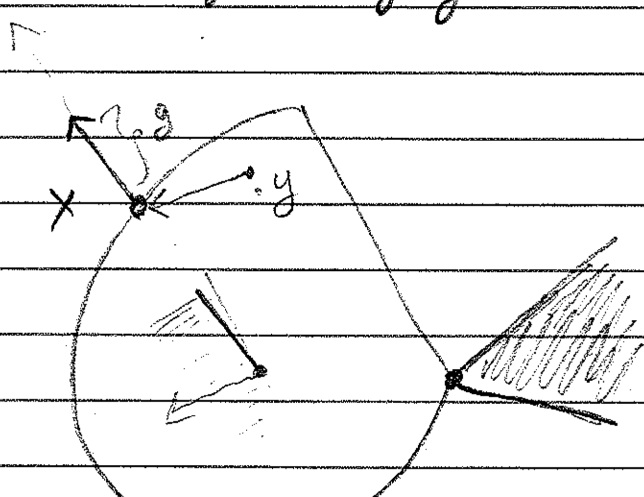
$$I_C(y) \geq \underbrace{I_C(x)}_{0} + g^T(y-x)$$

$y \notin C$: LHS $= \infty$ ✓

$y \in C$: LHS $= 0$

$$0 \geq g^T(y-x)$$

$$g^T x \geq g^T y$$



$N_C(x)$ is a cone

$g^T(x-y) \geq 0$ all y

$\{g : g^T(x-y) \geq 0\}$

x which f_i's are active at x?

i.e., $f_i(x) = f(x)$

$$\partial f(x) = \text{conv}\left(\bigcup_{\substack{\text{active} \\ f_i\text{'s}}} \partial f_i(x) \right)$$

---

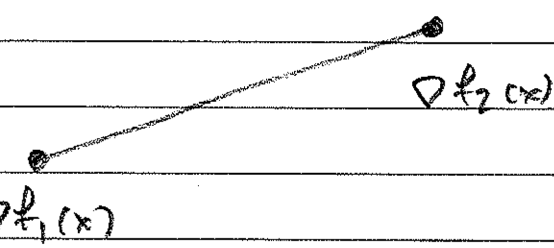
e.g. $f(x) = \max\{f_1(x), f_2(x)\}$

diff.

x s.t. $f_1(x) > f_2(x)$, $\quad \partial f(x) = \{\nabla f_1(x)\}$

same for $f_2(x) > f_1(x)$

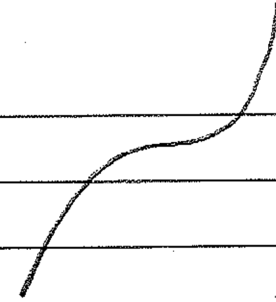
x s.t. $f_1(x) = f_2(x)$ $\quad \partial f(x) = \text{conv}\{\nabla f_1(x), \nabla f_2(x)\}$



$$\|x\|_p = \max_{\substack{y \text{ s.t.} \\ \|y\|_q \le 1}} y^T x \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

$$f(x) = \max_{y \in S} y^T x$$

$$\partial \|x\|_p = \left\{ y : \|y\|_q \le 1 \text{ and } y^T x = \max_{\|z\|_q \le 1} z^T x \right\}$$
$$= \operatorname{argmax}_{\|y\|_q \le 1} y^T x$$



$$\min_{x \in \mathbb{R}^n} f(x) = f(x^*)$$

$$\iff 0 \in \partial f(x^*)$$

g s.g. of f at x means

$\rightarrow$ $f(y) \geq f(x) + g^T(y-x)$ all $y \in \mathbb{R}^n$

if $g=0$, we get

$f(y) \geq f(x)$ all $y \in \mathbb{R}^n$

$$\min_{x \in C} f(x) \qquad C = \{x : g_i(x) \leq 0,\ h_j(x) = 0\}$$

$$= \min_{x \in \mathbb{R}^n} \underbrace{f(x) + I_C(x)}_{g(x)}$$

x^* is a minimizer $\iff 0 \in \partial g(x^*)$

$\iff 0 \in \partial f(x^*) + \mathcal{N}_C(x^*)$

$\iff -v \in \mathcal{N}_C(x^*)$

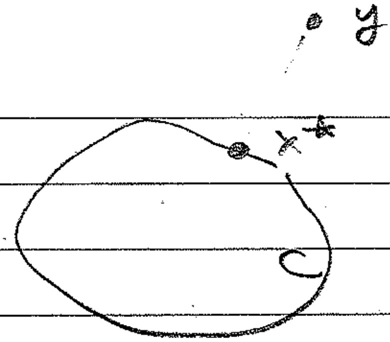
for some $v \in \partial f(x^*)$

for diff. f

$\iff -\nabla f(x^*) \in \mathcal{N}_C(x^*)$

$\iff \nabla f(x^*)^T (y - x^*) \geq 0$

all $y \in C$



$$\min_{x \in C} \|y-x\|_2$$

$$\iff \min_{x \in \mathbb{R}^n} \underbrace{\tfrac{1}{2}\|y-x\|_2^2}_{f(x)} + I_C(x)$$

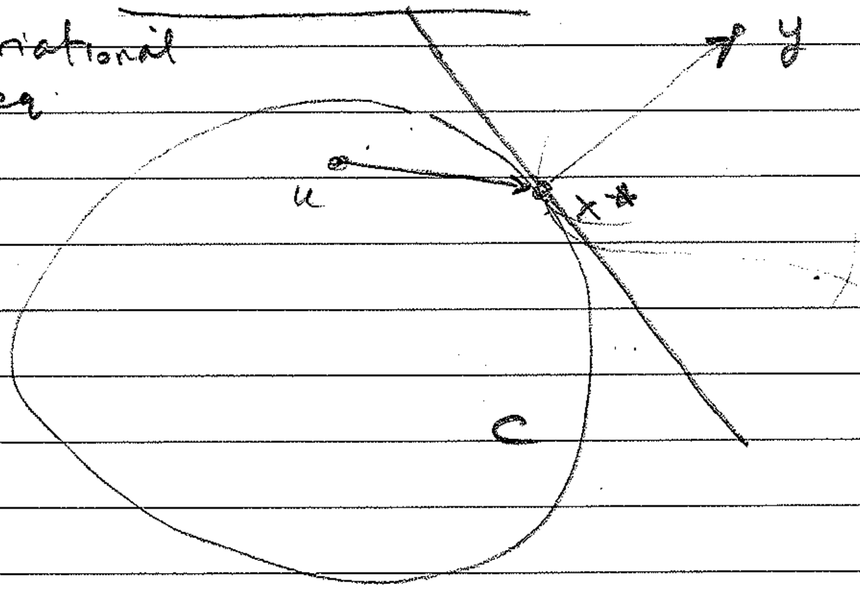
$$\nabla f(x) = x - y$$

x^* is opt. $\iff y - x^* \in N_C(x^*)$

$\iff (y-x^*)^T x^* \geq (y-x^*)^T u$ for all $u \in C$

$\iff \underline{(y-x^*)^T (x^* - u) \geq 0}$ for all $u \in C$

variational ineq



$\Sigma |\beta_j|$

$\shortparallel$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \ \frac{1}{2} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1$$

$$X = I$$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \ \frac{1}{2} \|y - \beta\|_2^2 + \lambda \|\beta\|_1 \Longleftarrow$$

st. conv $f(\alpha x + (1-\alpha) y)$

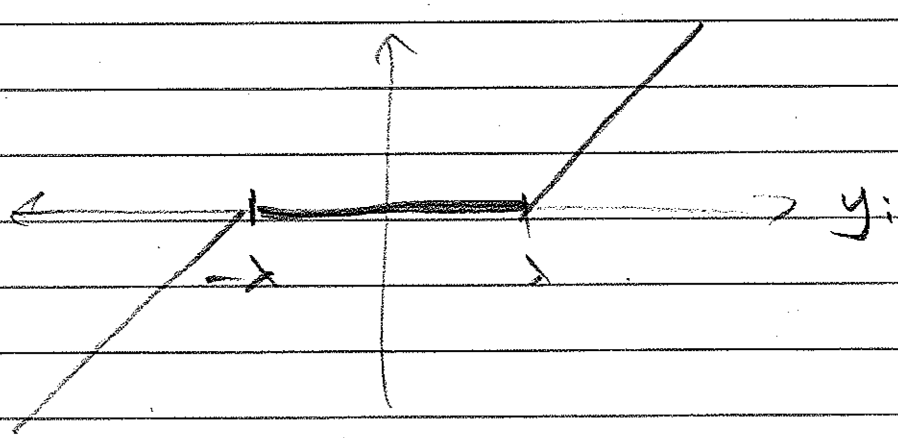
$< \alpha f(x) + (1-\alpha) f(y)$

$\min_x \ C$

$\min_\beta \|y - X\beta\|_2^2$

$$\hat{\beta} = S_\lambda(y)$$

$$\hat{\beta}_i = \begin{cases} y_i - \lambda & y_i > \lambda \\ 0 & y_i \in [-\lambda, \lambda] \\ y_i + \lambda & y < -\lambda \end{cases}$$



$$f(\beta) = \frac{1}{2}\|y-\beta\|_2^2 + \lambda\|\beta\|_1$$

$$\partial f(\beta) \ni \beta - y + \lambda \cdot v$$

$$v_i = \operatorname{sign}(\beta_i) \qquad \beta_i \neq 0$$
$$\in [-1,1] \qquad \beta_i = 0.$$

$$\beta = S_\lambda(y)$$

if: $y_i > \lambda$ $\qquad sg = \cancel{y_i} - \lambda - \cancel{y_i} + \lambda \cdot \cancel{v_i}^{\,1}$

$\beta_i = y_i - \lambda$ $\qquad = 0.$

✓

if $y_i < -\lambda$

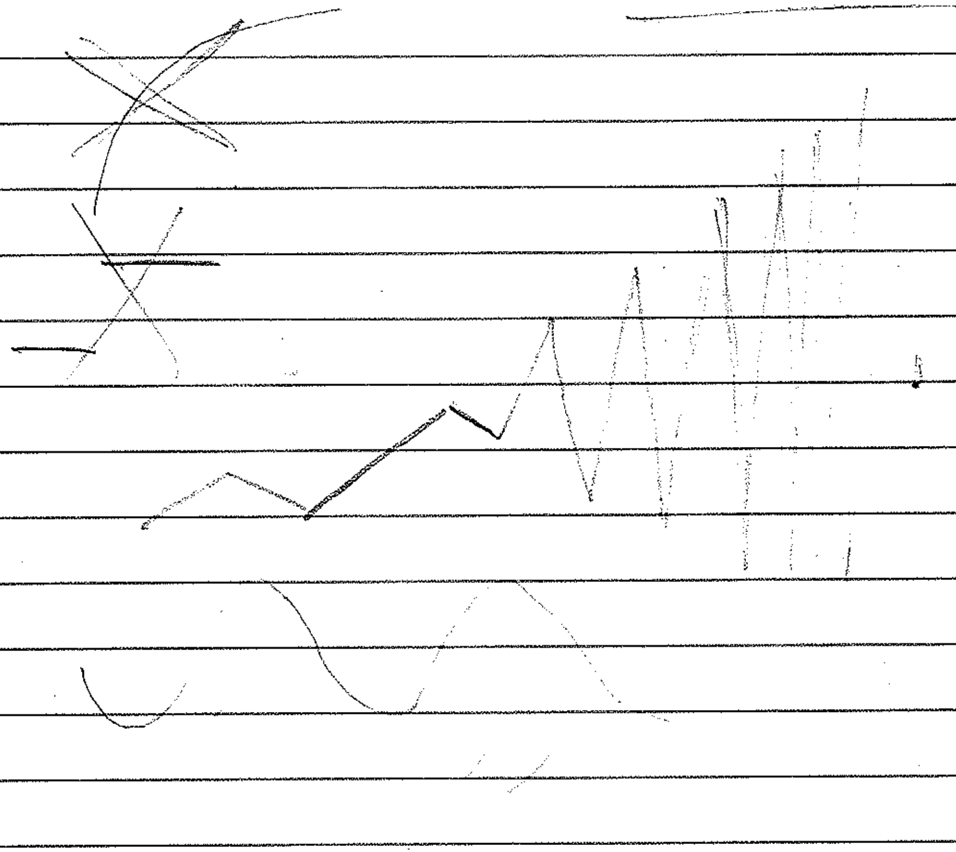
if $y_i \in [-\lambda, \lambda]$ $\qquad sg = 0 - y_i + \lambda \cdot v_i$

$\beta_i = 0.$

$$v_i = y_i/\lambda \in [-1,1]$$
$$= 0.$$

$t_k = \frac{1}{k}$

f Lipschitz



$\{x : \|x^* - x\|_2 \le R\} = S$