x feas $\qquad u \geq 0,\ v$

$$f(x) - g(u,v)$$

$$f(x) - f^* \leq f(x) - g(u,v) \quad \Leftarrow$$
$$\leq \varepsilon$$

$$f^* \geq g(u,v)$$

if $\quad f(x) - g(u,v) = 0$

$\Rightarrow \quad x$ is primal optimal
$\quad u,v$ are dual optimal

$$L(x,u,v) = f(x) + \sum u_i h_i(x) + \sum v_j \ell_j(x)$$

$$0 \in \partial_x L(x,u,v)$$

$$L(x,u,v) = \min_{\tilde{x}} L(\tilde{x},u,v)$$

Necessity

assume x^*, u_i^* v^* optimal and strong duality holds $\Longrightarrow$ KKT conditions must hold.

$$f(x^*) = g(u^*, v^*)$$

$$= \min_{x \in \mathbb{R}^n} f(x) + \sum u_i^* h_i(x) + \sum v_j^* l_j(x)$$

$$= \;\le\; f(x^*) + \sum \underbrace{u_i^* h_i(x^*)}_{\le 0} + \sum v_j^* \underbrace{l_j(x^*)}_{=0}$$

$$= \;\le\; f(x^*)$$

so all inequalities are actually equalities

x^* minimizes $L(x, u^*, v^*)$, i.e. stationarity

also $u_i^* h_i(x^*) = 0$ all i, i.e. complementary slackness

Sufficiency

x^*, u^*, v^* satisfy KKT conditions $\Longrightarrow$ x^*, u^*, v^* optimal

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$$g(u^*, v^*) = \min_x L(x; u^*, v^*)$$

$$= f(x^*) + \sum u_i^* h_i(x^*) + \sum v_j^* l_j(x^*)$$

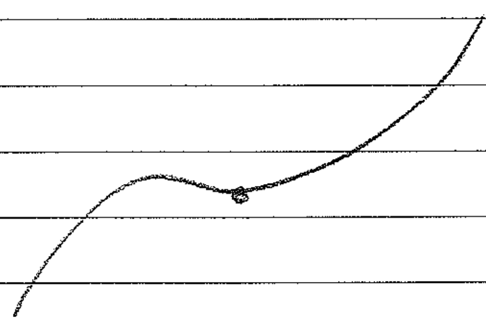
(stationarity)

$$= f(x^*)$$

duality gap is zero at x^*, u^*, v^*.
$\Rightarrow$ optimal

$$0 = \nabla f(x) + \sum u_i \nabla h_i(x) + \sum v_j \nabla l_j(x)$$

Same as stationarity as long f, h_i, l_j are convex



$$f(x) + \sum \mathbb{1}\{h_i(x) \le 0\} + \sum \mathbb{1}\{\ell_j(x) = 0\}$$

$$0 \in \partial f(x^*) + \sum \mathcal{N}_{\{h_i(x) \le 0\}}(x^*) + \sum \mathcal{N}_{\{\ell_j(x) = 0\}}(x^*)$$

$\iff x^*$ opt.

$$\min \quad \tfrac{1}{2} x^T Q x + c^T x \qquad x^* = -Q^{-1} c$$

$$\text{st.} \quad Ax = 0$$

Newton's $\min f(x)$ st. $Ax = b$

$$x^+ = x - (\nabla^2 f(x))^{-1} \nabla f(x)$$

$$Ax = b$$

$$x^+ = x + \Delta \quad \text{where } A\Delta = 0.$$

and $\quad \tfrac{1}{2} \Delta^T (\nabla^2 f(x)) \Delta + \nabla f(x)^T \Delta$

$$L(x, u) = \tfrac{1}{2} x^T Q x + c^T x + u^T A x$$

stat. $\quad Qx + c + A^T u = 0.$

comp. slackers $\quad \emptyset$

p. feas $\quad Ax = 0.$

d. feas $\quad \emptyset$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}$$

$$\min -\sum \log(\alpha_i + x_i)$$

$$\text{s.t.} \quad x \succeq 0, \quad \mathbf{1}^T x = 1$$

stat. $L(x,u,v) = -\sum \log(\alpha_i + x_i) - \sum u_i x_i + v(\sum x_i - 1)$

$$\frac{-1}{\alpha_i + x_i} - u_i + v = 0 \qquad i = 1, \ldots n$$

c.s. $u_i x_i = 0 \qquad i = 1 .. n$

p.f. $x \succeq 0, \ \mathbf{1}^T x = 1$

d.f. $u \succeq 0$.

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stat/d.f. $v - \dfrac{1}{\alpha_i + x_i} \geq 0$

c.s. $x_i \left(v - \dfrac{1}{\alpha_i + x_i} \right) = 0 \qquad i = 1 \ldots n$

p.f. $x \succeq 0, \ \mathbf{1}^T x = 1$.

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if $v < \dfrac{1}{\alpha_i}$ $x_i \neq 0$, $x_i > 0$.

then $v = \dfrac{1}{\alpha_i + x_i}$, i.e. $x_i = \dfrac{1}{v} - \alpha_i$

if $v \geq \dfrac{1}{\alpha_i}$ $x_i \not> 0$, $x_i = 0$.

$$x_i = \max\left\{0, \frac{1}{v} - \alpha_i\right\}$$

$$1 = \sum x_i = \sum \max\{0, \frac{1}{v_i} - \alpha_i\}$$