Lasso

$$\min_{\beta} \frac{1}{2}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1$$

$$0 = -X^T(y - X\beta) + \lambda s$$

$$s_i = \begin{cases} \text{sign}(\beta_i) & \text{if } \beta_i \neq 0 \\ \in [-1, 1] & \text{if } \beta_i = 0. \end{cases}$$

$$X_i^T(y - X\beta) = \lambda s_i \qquad i = 1, \ldots p$$

if $|X_i^T(y - X\beta)| < \lambda \Rightarrow \beta_i = 0.$

$|X_i^T(y - X\beta)| = \lambda \Rightarrow s_i = \pm 1$

$\cancel{\Rightarrow} \ \beta_i \neq 0$

Group lasso

$$\min_{\beta = (\beta^{(1)}, \beta^{(2)}, \ldots \beta^{(G)})} \left\{ \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \sum_{i=1}^{G} \omega_i \|\beta^{(i)}\|_2 \right\}$$

$$0 = -(X^{(i)})^T(y - X\beta) + \lambda \omega_i s^{(i)}$$

$i = 1, \ldots G$ where $s^{(i)} \in \partial \|\beta^{(i)}\|_2$

i.e., $s_i = \begin{cases} \beta^{(i)} / \|\beta^{(i)}\|_2 & \text{when } \beta^{(i)} \neq 0 \\ \in \{v : \|v\|_2 \leq 1\} & \text{when } \beta^{(i)} = 0 \end{cases}$

$$(X^{(i)})^T (y - X\beta) = \lambda w_i s^{(i)}$$

if $\|(X^{(i)})^T(y - X\beta)\|_2 < \lambda w_i$

$$\Rightarrow \beta^{(i)} = 0$$

if $\beta^{(i)} \neq 0$

$$(X^{(i)})^T \left(y - X^{(i)}\beta^{(i)} - \sum_{j \neq i} X^{(j)}\beta^{(j)}\right) = \lambda w_i \frac{\beta^{(i)}}{\|\beta^{(i)}\|_2}$$

$$-X^{(i)T} X^{(i)} \beta^{(i)} + X^{(i)T} r^{(i)} = \lambda w_i \frac{\beta^{(i)}}{\|\beta^{(i)}\|_2}$$

$$\left(\frac{\lambda w_i}{\|\beta^{(i)}\|_2} I + X^{(i)T} X^{(i)}\right) \beta^{(i)} = X^{(i)T} r^{(i)}$$

$$\beta^{(i)} = \left(X^{(i)T} X^{(i)} + \frac{\lambda w_i}{\|\beta^{(i)}\|_2}\right)^{-1} X^{(i)T} r^{(i)}$$