u^*, v^* dual opt.

x^* must minimize $\underbrace{L(x, u^*, v^*)}$ over x

$$0 \in \partial f(x) + \sum u_i^* h_i(x) + \sum v_j^* l_j(x)$$

$$\min_x \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t. } a^T x = b$$

f_i strictly convex, diff

$$L(x, v) = \sum_{i=1}^{n} f_i(x_i) + v\left(b - \sum a_i x_i\right)$$

$$g(v) = \min_x L(x, v)$$

$$= \sum \underbrace{\min_{x_i} \left(f_i(x_i) - v a_i x_i\right)} + vb$$

$$= \sum -f_i^*(a_i v) + vb$$

$$\max_v g(v) \Rightarrow v^*$$

x^* must minimize $L(x, v^*)$ over x

i.e. solve $\nabla f_i(x_i) = v^* a_i$ for each i

$$f_i(x_i) = \frac{(y_i - x_i)^2}{2}$$

$$x_i - y_i = v^* \cdot a_i$$
$$x_i^* = v^* a_i + y_i$$

---

norm $\quad \|x\| \qquad$ e.g. $\|\cdot\|_p, \ \|\cdot\|_{nuc}$

dual norm $\quad \|y\|_* = \max_{\|x\| \le 1} x^T y$

A, B

$\mathrm{tr}(A^T B)$

$\forall x, y \quad |x^T y| \le \|x\| \, \|y\|_*$

$$z = \frac{x}{\|x\|}, \quad \|z\| \le 1.$$

$$\|y\|_* = \max_{\|x\| \le 1} x^T y \ge z^T y$$
$$= \frac{x^T y}{\|x\|}$$

$$\Rightarrow x^T y \le \|x\| \, \|y\|_*$$

---

duals: $\quad (\|\cdot\|_p)_* = \|\cdot\|_q \ \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$

$$(\|\cdot\|_{nuc})_* = \|\cdot\|_{op}$$

$$\|\cdot\|_{**} = \|\cdot\|$$

$f(x)$

$$f^*(y) = \max_x \{x^T y - f(x)\}$$

---

f^* is always convex.
(true even if f is not convex)

$\forall\ x, y \qquad f(x) + f^*(y) \geq x^T y$

$f^{**} \leq f$
$f^{**} = f \qquad$ if f closed and convex

$f(x) = \frac{1}{2} x^T Q x \qquad Q \succ 0. \quad$ symm.

$$f^*(y) = \max_x \ \underbrace{x^T y - \frac{1}{2} x^T Q x}$$

$x = Q^{-1} y$

$= y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} \cancel{Q Q^{-1}} y$

$= \frac{1}{2} y^T Q^{-1} y$

$f(x) + f^*(y) \geq x^T y$

$\frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y \geq x^T y$

$$f(x) = I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

$$f^*(y) = \max_x \; x^T y - I_C(x)$$

$$= \max_{x \in C} \; x^T y$$ support function

$$f(x) = \|x\|$$

$$f^*(y) = I_{\{z : \|z\|_* \le 1\}}(y)$$ ✓

$$f^{**} = f$$

$$\left(\max_{x \in C} \; x^T y \right)^* = I_C(x)$$

$$\|x\| = \max_{\|z\|_* \le 1} \; x^T z$$

$$\min_{\beta} \overbrace{\tfrac{1}{2}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1}^{f(x)}$$

$$L(x) =$$

$$\cancel{g(\phi) = \min_{x} L(x) = f^*}$$

$$\min_{\beta, z} \tfrac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 \quad \text{s.t. } z = X\beta$$

$$L(\beta, z, u) = \tfrac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$$

$$= \underbrace{\tfrac{1}{2}\|y - z\|_2^2 + u^T z}_{\min z} + \lambda\underbrace{\left(\|\beta\|_1 - \left(\frac{X^T u}{\lambda}\right)^T \beta\right)}_{\min \beta}$$

$$= \tfrac{1}{2}\|y\|_2^2 - \tfrac{1}{2}\|y - u\|_2^2$$

$$- \lambda \cdot I_{\{v : \|v\|_\infty \le 1\}}\left(\frac{X^T u}{\lambda}\right)$$

$$g(u) = \min_{\beta, z} L(\beta, z, u)$$

$$= \begin{cases} \tfrac{1}{2}\|y\|_2^2 - \tfrac{1}{2}\|y - u\|_2^2 & \text{when } \|X^T u\|_\infty \le \lambda \\ -\infty & \text{else} \end{cases}$$

Dual.

$$\boxed{\begin{aligned} &\max_{u} \ \tfrac{1}{2}\|y\|_2^2 - \tfrac{1}{2}\|y - u\|_2^2 \longleftarrow g(u) \\ &\text{s.t. } \|X^T u\|_\infty \le \lambda \end{aligned}}$$

Dual $\Longleftrightarrow$

$$\min \ \|y-u\|_2^2 \quad \leftarrow \tilde{g}$$
$$\text{s.t.} \ \|X^T u\|_\infty \le \lambda$$

"Dual"

$$f(\hat{\beta}) = g(\hat{u})$$
$$\neq \tilde{g}(\hat{u})$$

$$\frac{1}{2}\|y - Xz\|_2^2 + \lambda\|\beta\|_1 \quad \text{s.t. } z = \beta$$