$$\|y\|_* = \max_{\|z\| \le 1} z^T y$$

$$\|y\| = (\|y\|_*)_* = \max_{\|z\|_* \le 1} z^T y$$

$$\left(1_{\{z : \|z\|_* \le 1\}}\right)^*(y) = \max_{z : \|z\|_* \le 1} z^T y$$

$$= \|y\|$$

$$1_{\{z : \|z\|_* \le 1\}} = (\|y\|)^*$$

X $n \times p$

(P) $$\min_{\beta} \frac{1}{2}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1$$ $\boxed{p \text{ vars}}$

$$\min_{z,\beta} \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 \quad \text{s.t. } z = X\beta$$

$\boxed{n \text{ vars}}$

(d) $$\max_{u} \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2 \quad \text{s.t. } \|X^T u\|_\infty \le \lambda$$

$$L(\beta, z, u) = \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$$

$L(\beta, z, \hat{u})$ minimized at $\hat{\beta}, \hat{z}$

$$\nabla_z L = z - y + \hat{u} = 0$$

$$\hat{z} = y - \hat{u}, \text{ i.e. } \boxed{X\hat{\beta} = y - \hat{u}}$$

$$f^*(y) = \max_x \; x^T y - f(x)$$

$$f^*(y) = -\min_x \; f(x) - x^T y$$

$$\boxed{-f^*(y) = \min_x \; f(x) - x^T y}$$

(p) $\min_x \; f(x) + g(x)$

$\iff \min_{x,z} \; f(x) + g(z) \quad \text{st} \quad x = z$

$$L(x, z, u) = f(x) + g(z) + u^T (z - x)$$

$$g(u) = \min_{x,z} L(x, z, u)$$

$$= \underbrace{\min_x \; f(x) - u^T x}_{} + \underbrace{\min_z \; g(z) + u^T z}_{}$$

$$= -f^*(u) \quad - g^*(-u)$$

(d) $\max_u \; -f^*(u) - g^*(-u)$

$$-f^*(-u) - g^*(u)$$

$$f^*(y) = \max \; x^T y - f(x)$$