10-725: Optimization **Fall 2013**

Lecture 16: Penalty Methods, October 17

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16.1 Penalty Methods

16.1.1 Problem Setup

Many times we have the constrained optmization problem (**P**):

$$\min_{x \in \mathcal{S}} f(x)$$

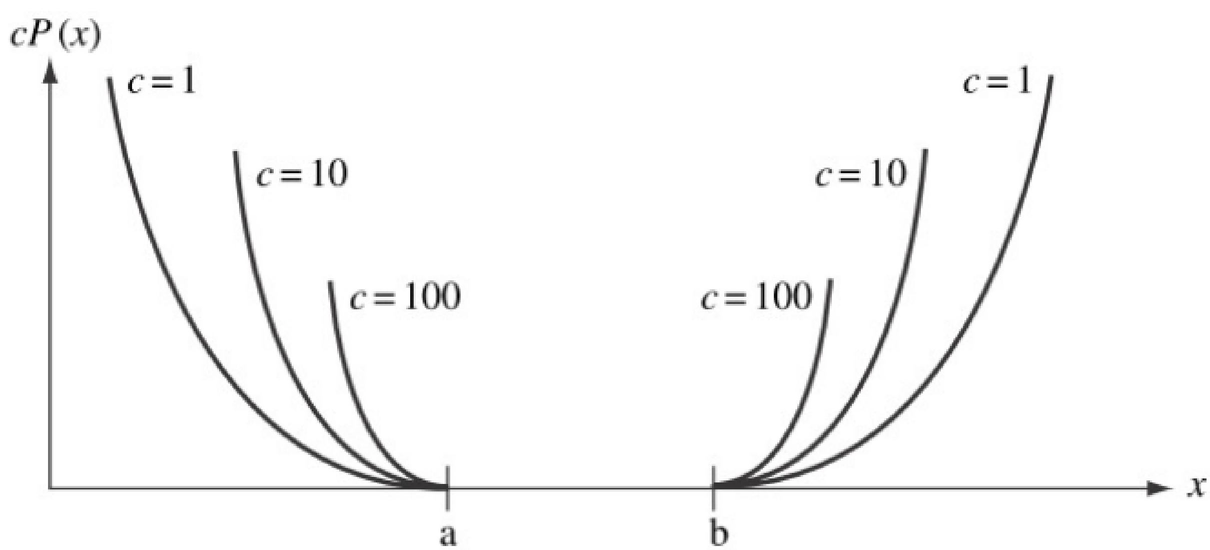
where $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\mathcal{S}$ is a constraint set in $\mathbb{R}^n$.

We introduce the **Penalty program**, ($\mathbf{P}(\mathbf{c})$), the unconstrained problem:

$$\min_{x \in \mathbb{R}^n} f(x) + cp(x)$$

where $c > 0$ and $p : \mathbb{R}^n \to \mathbb{R}$ is the **penalty function** where $p(x) \geq 0 \; \forall \; x \in \mathbb{R}^n$, and $p(x) = 0$ iff $x \in S$.

Intuitively, the **penalty term** is used to give a high cost for violation of the constraints.



16.1.2 Inequality and Equality Constraints

For example, if we are given a set of inequality constraints (i.e. $\mathcal{S} = \{x : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$), a useful penalty function could be $p(x) = \frac{1}{2}\sum_{i=1}^{m} (\max[0, g_i(x)])^2$. That is, if we satisfy the constraint, we don't take any penalty. Otherwise we take a squared penalty. Depending on c, we weight this penalty in $(P(c))$. For equality constraints we can rewrite them as inequality constraints and use them as above. That is, rewrite $h_j(x) = 0$ as two inequality constraints, $h_j(x) \leq 0$ and $-h_j(x) \leq 0$.

For large c, the minimum point of a problem $(P(c))$ is in a region where the penalty p is small. In fact, we will prove below that as $c \to \infty$, the solution of the penalty problem $(P(c))$ will converge to a solution of the constrained problem (P).

16.2 Penalty Method Lemmas

Let $0 < c_1 < c_2 < \ldots < c_k < c_{k+1} < \ldots \to \infty$ be our penalty parameter. Let $q(c, k) := f(x) + cp(x)$ be our penalty program. Also, let $x_k = \arg\min_x q(c_k, x) = \arg\min_x f(x) + c_k p(x)$.

With this notation, we will prove the following for **penalty lemmas**:

1. $q(c_k, x_k) \leq q(c_{k+1}, x_{k+1})$
2. $p(x_k) \geq p(x_{k+1})$
3. $f(x_k) \leq f(x_{k+1})$
4. $f(x^*) \geq q(c_k, x_k) \geq f(x_k)$

Below, we provide proofs of each of the above lemmas.

Lemma 16.1 $q(c_k, x_k) \leq q(c_{k+1}, x_{k+1})$

Proof:

$$\begin{aligned}
q(c_{k+1}, x_{k+1}) &= f(x_{k+1}) + c_{k+1}p(x_{k+1}) \\
&\geq f(x_{k+1}) + c_k p(x_{k+1}) && (\because c_{k+1} > c_k > 0) \\
&\geq f(x_k) + c_k p(x_{k+1}) && (\because x_k \text{ is the minimizer of } q(c_k, x)) \\
\boxed{\therefore q(c_{k+1}, x_{k+1}) \geq q(c_k, x_k)} & && (\because q(c, x_{k+1} = f(x_k) + c_k p(x_{k+1}))
\end{aligned}$$

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Lemma 16.2 $p(x_k) \geq p(x_{k+1})$

Proof:

$$f(x_k) + c_k p(x_k) \leq f(x_{k+1} + c_k p(x_{k+1}) \qquad (\because x_k \text{ is the minimizer of } q(c_k, x)) \tag{16.1}$$

$$f(x_{k+1}) + c_{k+1}p(x_{k+1}) \leq f(x_k) + c_{k+1}p(x_k) \qquad (\because x_{k+1} \text{ is the minimizer of } q(c_{k+1}, x)) \tag{16.2}$$

Adding Equation 16.1 and Equation 16.2 together, we get

$$c_k p(x_k) + c_{k_+} p(x_{k+1}) \leq c_k p(x_{k+1}) + c_{k+1} p(x_k)$$
$$\Rightarrow (c_{k+1} - c_k) p(x_{k+1}) \leq (c_{k+1} - c_k) p(x)$$
$$\boxed{\therefore p(x_{k+1}) \leq p(x_k)} \qquad (\because c_{k+1} > c_k \Rightarrow c_{k+1} - c_k > 0)$$

■

Lemma 16.3 $f(x_k) \leq f(x_{k+1})$

Proof:

$$\begin{aligned} f(x_{k+1}) + c_k p(x_{k+1}) &\geq f(x_k) + c_k p(x_k) && (\because x_k \text{ is the minimizer of } q(c_k, x)) \\ &\geq f(x_k) + c_k p(x_{k+1}) && (\because \text{Lemma 16.2}) \end{aligned}$$
$$\boxed{\therefore f(x_{k+1}) \geq f(x_k)}$$

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Lemma 16.4 *Let x^* be the optimal value of our original constrained problem (P) with constraint set $\mathcal{S}$. Then, $f(x^*) \geq q(c_{k+1}, x_{k+1}) \geq f(x_k)$ $\forall k$.*

Proof:

$$\begin{aligned} f(x^*) &= f(x^*) + c_k p(x^*) && (\because x^* \in \mathcal{S} \Rightarrow p(x^*) = 0) \\ &\geq f(x_k) + c_k p(x_k) \geq f(x_k) && (\because x_k \text{ is the minimizer of } q(c_k, x), \text{ and } c_k > 0,\ p(x_k) \geq 0) \end{aligned}$$
$$\boxed{\therefore f(x^*) \geq q(c_{k+1}, x_{k+1}) \geq f(x_k)\ \forall k}$$

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16.3 Convergence of the Penalty Method

Using the lemmas developed in Section 16.2, we prove the Penalty convergence theorem.

Theorem 16.5 *Suppose f, g, p are continuous functions. Let $x_k = \arg\min_x f(x) + c_k p(x)$ for a penalty function $p(x)$ as defined in subsection 16.1.1. Let $0 < c_1 < c_2 < \ldots < c_k < c_{k+1} < \ldots \to \infty$. Let $\bar{x}$ be an arbitrary limit point of $\{x_k\}_{k=1}^{\infty}$.*

Then, $\bar{x}$ solves (P) where (P) is the original constrained problem $\min_x f(x)$ s.t. $g(x) \leq 0$.

Proof: The limit point is defined as $\bar{x} = \lim_{k \in \mathcal{K}} x_k$.

Since f is given as continuous, then $\lim_{k \in \mathcal{K}} f(x_k) = f(\bar{x})$. We then get,

$$\begin{aligned}
q^* := \lim_{x \in \mathcal{K}} q(c_k, x_k) &\leq f(x^*) \qquad (\because \text{Lemma 16.4}) \\
\Rightarrow q^* = \lim_{x \in \mathcal{K}} f(x_k) + \lim_{x \in \mathcal{K}} c_k p(x_k) &\leq f(x^*) \\
\Rightarrow q^* = f(\bar{x}) + \lim_{x \in \mathcal{K}} c_k p(x_k) &\leq f(x^*) \\
\Rightarrow q^* - f(\bar{x}) = \lim_{x \in \mathcal{K}} c_k p(x_k) &\leq f(x^*)
\end{aligned}$$

Since $q^* - f(\bar{x})$ and $f(x^*)$ are finite which means $\lim_{x \in \mathcal{K}} c_k p(x_k)$ has to be a finite quantity. Since we know that $c_k \to \infty$, $p(x_k) \to 0$. This means that $p(\bar{x}) = 0$, which from the definition of p tells us that $\bar{x} \in S$ where S is our constraint set. ■

16.4 Frequently used penalty functions

1. **Polynomial penalty:** $p(x) = \sum_{i=1}^{m} [\max\{0, g_i(x)\}]^q, q \geq 1$

   (a) Linear penalty: $(q = 1) : p(x) = \sum_{i=1}^{m} [\max\{0, g_i(x)\}]$

   (b) Quadratic penalty: $(q = 2) : p(x) = \sum_{i=1}^{m} [\max\{0, g_i(x)\}]$

   For example, if we define $g_i^+(x) = \max\{0, g_i(x)\}$, then $g^+(x) = [g_1^+(x), .., g_m^+(x)]^T$. The penalty function $P(x) = g^+(x)^T g^+(x)$, or $P(x) = g^+(x)^T \Gamma g^+(x)$ where $\Gamma > 0$

2. **Penalty for problem with equality and inequality constraints**

$$\begin{aligned}
&P : \min_x f(x) \\
&\text{s.t. } g(x) \leq 0 \\
&\quad\;\; h(x) = 0 \\
&\quad\;\; x \in \mathbb{R}^n \\
\text{Need penalty function: } &p(x) = 0 \text{ if } g(x) \leq 0 \text{ AND } h(x) = 0 \\
&p(x) > 0 \text{ if } g(x) > 0 \text{ OR } h(x) \neq 0 \\
\text{We can use: } &p(x) = \sum_{i=1}^{m} [\max\{0, g_i(x)\}]^q + \sum_{i=1}^{k} |h_i(x)|^q, q \geq 1
\end{aligned}$$

16.5 Derivative of the penalty function

Suppose we use $P(x) = \gamma(g^+(x))$, where $g^+(x)$ is as defined previously. An example of $\gamma(x)$ is $\gamma(x) = y^T y$. The difficulty arises when we try to take the derivative of $P(x)$, as the max function $g^+(x)$ is not differentiable. But we will see that if we choose $\gamma(x)$ appropriately, we can make $P(x)$ differentiable.

$$\begin{aligned}
\frac{\partial P(x)}{\partial x} &= \sum_{i=1}^{m} \frac{\partial \gamma(g^+(x))}{\partial (g_i^+(x))} \frac{\partial g_i^+(x)}{\partial x} \\
\frac{\partial g^+(x)}{\partial x} &= \begin{cases} \frac{\partial g_i(x)}{\partial x} \text{ if } g_i(x) \geq 0 \\ 0 \text{ if } g_i(x) < 0 \end{cases}
\end{aligned}$$

But $\frac{\partial g_i^+(x)}{\partial x}$ may not be continuous at 0. However, if we choose γ such that $\frac{\partial \gamma(g^+(x))}{\partial y_i} = 0$ whenever $g_i(x) = 0$, then it won't matter if $\frac{\partial g_i^+(x)}{\partial x}$ is discontinuous, because it will be multiplied by 0. One such $\gamma(x)$ is $\sum_{i=1}^m [g_i^+(x)]^q, q \geq 1$

16.6 KKT in penalty methods

As before, we have:

1. Penalty program: $x_k = \arg\min_x f(x) + c_k P(x)$

2. Penalty function: $P(x) = \gamma(g^+(x))$

3. Derivatives: $\nabla P(x) = \sum_{i=1}^m \frac{\partial \gamma(g^+(x))}{\partial (g_i^+(x))} \frac{\partial g_i^+(x)}{\partial x}$

The 1st order condition in local minimum tells us:

$$0 = \nabla f(x_k) + c_k \nabla P(x_k) = \nabla f(x_k) + \sum_{i=1}^m u_{i,k} \nabla g_i(x_k) \text{ where } u_{i,k} = c_k \frac{\partial \gamma(g^+(x_k))}{\partial (g_i^+(x_k))}$$
$$0 = \nabla f(x_k) + (u_k)^T \nabla g(x_k)$$

u_k now looks like a Lagrange multiplier. Indeed, under some mild conditions, as $x_k \to x^* \implies u_k \to u^*$, where u^* is the Lagrange multiplier at the optimum.