Convexity II: Optimization Basics

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Convex Optimization 10-725/36-725

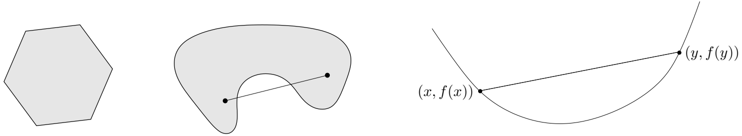
See supplements for reviews of

- *basic multivariate calculus*
- *basic linear algebra*

Last time: convex sets and functions

"Convex calculus" makes it easy to check convexity. Tools:

- Definitions of convex sets and functions, classic examples



- Key properties (e.g., first- and second-order characterizations for functions)
- Operations that preserve convexity (e.g., affine composition)

E.g., is $\max\left\{\log\left(\frac{1}{(a^T x + b)^7}\right), \|Ax + b\|_1^3\right\}$ convex?

Outline

Today:

- Optimization terminology
- Properties and first-order optimality
- Equivalent transformations

Optimization terminology

Reminder: a convex optimization problem (or program) is

$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \; i = 1, \ldots m \\ & Ax = b \end{aligned}$$

where f and g_i, $i = 1, \ldots m$ are all convex, and the optimization domain is $D = \mathrm{dom}(f) \cap \bigcap_{i=1}^m \mathrm{dom}(g_i)$ (often we do not write D)

- f is called criterion or objective function
- g_i is called inequality constraint function
- If $x \in D$, $g_i(x) \leq 0$, $i = 1, \ldots m$, and $Ax = b$ then x is called a feasible point
- The minimum of $f(x)$ over all feasible points x is called the optimal value, written $f^\star$

- If x is feasible and $f(x) = f^\star$, then x is called an optimal point, a solution, or a minimizer[1]
- If x is feasible and $f(x) \leq f^\star + \epsilon$, then x is called ϵ-suboptimal
- If x is feasible and $g_i(x) = 0$, then we say g_i is active at x
- Convex minimization can be reposed as concave maximization

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \\ & \quad i = 1, \ldots m \\ & Ax = b \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \max & -f(x) \\ \text{s.t.} & g_i(x) \leq 0, \\ & \quad i = 1, \ldots m \\ & Ax = b \end{array}$$

Both are called convex optimization problems

[1] Note: a convex optimization problem need not have solutions, i.e., need not attain its minimum, but we will not be careful about this

Convex solution sets

Let X_{opt} be the set of all solutions of convex problem, written

$$\begin{array}{rcll} X_{\mathsf{opt}} & = & \text{argmin} & f(x) \\ & & \text{subject to} & g_i(x) \leq 0, \ i = 1, \ldots m \\ & & & Ax = b \end{array}$$

Key property: X_{opt} is a convex set

Proof: use definitions. If x, y are solutions, then for $0 \leq \theta \leq 1$,

- $\theta x + (1-\theta)y \in D$
- $g_i(\theta x + (1-\theta)y) \leq \theta g_i(x) + (1-\theta)g_i(y) \leq 0$
- $A(\theta x + (1-\theta)y) = \theta Ax + (1-\theta)Ay = b$
- $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) = f^\star$

Therefore $\theta x + (1-\theta)y$ is also a solution

Another key property: if criterion f is strictly convex, then the solution is unique, i.e., X_{opt} contains one element

Example: lasso

Given $y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times p}$, consider the lasso problem:

$$\begin{aligned} \min_{\beta \in \mathbb{R}^p} \quad & \|y - X\beta\|_2^2 \\ \text{subject to} \quad & \|\beta\|_1 \le s \end{aligned}$$

Is this a convex problem? What is the criterion function? What are the inequality constraints? Equality constraints?

What is the feasible set? Is the solution unique:

- when $n \ge p$ and X has full column rank?
- when $p > n$ ("high-dimensional" case)?

How do our answers change if we changed criterion to Huber loss:

$$\sum_{i=1}^n \rho(y_i - x_i^T \beta), \quad \rho(z) = \begin{cases} \frac{1}{2} z^2 & |z| \le \delta \\ \delta |z| - \frac{1}{2}\delta^2 & \text{else} \end{cases} \quad ?$$

Example: support vector machines

Given $y \in \{-1, 1\}^n$, $X \in \mathbb{R}^{n \times p}$ with rows $x_1, \ldots x_n$, consider the support vector machine or SVM problem:

$$\begin{aligned} \min_{\beta, \beta_0, \xi} \quad & \frac{1}{2}\|\beta\|_2^2 + C\sum_{i=1}^n \xi_i \\ \text{subject to} \quad & \xi_i \geq 0, \ i = 1, \ldots n \\ & y_i(x_i^T \beta + \beta_0) \geq 1 - \xi_i, \ i = 1, \ldots n \end{aligned}$$

Is this convex? What is the criterion, what are the constraints?

What is the feasible set? Is the solution (β, β_0, ξ) unique? What if changed the criterion to

$$\frac{1}{2}\|\beta\|_2^2 + \frac{1}{2}\beta_0^2 + C\sum_{i=1}^n \xi_i^2?$$

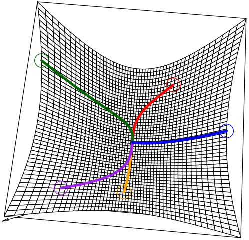
Local minima are global minima

For a convex problem, a feasible point x is called locally optimal is there is some $R > 0$ such that

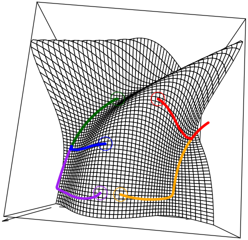
$$f(x) \le f(y) \quad \text{for all feasible } y \text{ such that } \|x - y\|_2 \le R$$

Reminder: for convex optimization problems, local optima are global optima

Proof simply follows from definitions



Convex



Nonconvex

Rewriting constraints

The optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \ i = 1, \dots m \\ & Ax = b \end{aligned}$$

can be rewritten as

$$\min \ f(x) \ \text{ subject to } \ x \in C$$

where $C = \{x : g_i(x) \leq 0, \ i = 1, \dots m, \ Ax = b\}$ is feasible set. Hence the above formulation is completely general

Note: we can further rewrite this in unconstrained form as

$$\min \ f(x) + I_C(x)$$

where I_C is the indicator function of C

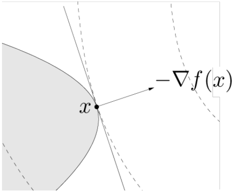
First-order optimality condition

For a convex problem

$$\min \ f(x) \ \text{ subject to } \ x \in C$$

and differentiable f, a feasible point x is optimal if and only if

$$\nabla f(x)^T (y - x) \geq 0 \quad \text{for all } y \in C$$



This is called the first-order condition for optimality

In words: all feasible directions from x are aligned with increasing gradient $\nabla f(x)$

Important special case: if $C = \mathbb{R}^n$ (unconstrained optimization), then optimality condition reduces to familiar $\nabla f(x) = 0$

Example: quadratic minimization

Consider minimizing the quadratic function

$$f(x) = \frac{1}{2}x^T Q x + b^T x + c$$

where $Q \succeq 0$. The first-order condition says that solution satisfies

$$\nabla f(x) = Qx + b = 0$$

Cases:

- if $Q \succ 0$, then there is a unique solution $x = Q^{-1}b$
- if Q is singular and $b \notin \operatorname{col}(Q)$, then there is no solution (i.e., $\min_x f(x) = -\infty$)
- if Q is singular and $b \in \operatorname{col}(Q)$, then there are infinitely many solutions

  $$x = Q^+ b + z, \quad z \in \operatorname{null}(Q)$$

  where Q^+ is the pseudoinverse of Q

Example: equality-constrained minimization

Consider the equality-constrained convex problem:

$$\min \; f(x) \;\; \text{subject to} \;\; Ax = b$$

with f differentiable. Let's prove Lagrange multiplier optimality condition

$$\nabla f(x) + A^T u = 0 \quad \text{for some } u$$

According to first-order optimality, solution x satisfies $Ax = b$ and

$$\nabla f(x)^T (y - x) \geq 0 \quad \text{for all } y \text{ such that } Ay = b$$

This is equivalent to

$$\nabla f(x)^T v = 0 \quad \text{for all } v \in \mathrm{null}(A)$$

Result follows since $\mathrm{null}(A)^\perp = \mathrm{row}(A)$

Example: projection onto a convex set

Consider projection onto convex set C:

$$\min \ \|a - x\|_2^2 \ \text{ subject to } \ x \in C$$

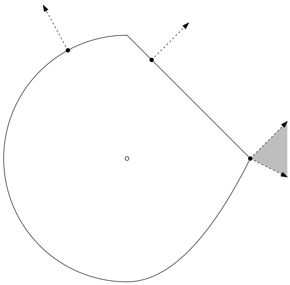
First-order optimality condition says that the solution x satisfies

$$\nabla f(x)^T(y - x) = (x - a)^T(y - x) \geq 0 \quad \text{for all } y \in C$$

Equivalently, this says that

$$a - x \in \mathcal{N}_C(x)$$

where recall $\mathcal{N}_C(x)$ is the normal cone to C at x



Partial optimization

Reminder: $g(x) = \min_{y \in C} f(x,y)$ is convex in x, provided that f is convex in (x,y) and C is a convex set

Therefore we can always partially optimize a convex problem and retain convexity

E.g., if we decompose $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$, then

$$\begin{array}{ll} \min\limits_{x_1,x_2} & f(x_1,x_2) \\ \text{s.t.} & g_1(x_1) \le 0 \\ & g_2(x_2) \le 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \min\limits_{x_1} & \tilde{f}(x_1) \\ \text{s.t.} & g_1(x_1) \le 0 \end{array}$$

where $\tilde{f}(x_1) = \min\{f(x_1,x_2) : g_2(x_2) \le 0\}$. The right problem is convex if the left problem is

Example: hinge form of SVMs

Recall the SVM problem

$$\begin{aligned} \min_{\beta,\beta_0,\xi} \quad & \frac{1}{2}\|\beta\|_2^2 + C\sum_{i=1}^n \xi_i \\ \text{subject to} \quad & \xi_i \geq 0,\ y_i(x_i^T\beta + \beta_0) \geq 1 - \xi_i,\ i = 1,\ldots n \end{aligned}$$

Rewrite the constraints as $\xi_i \geq \max\{0, 1 - y_i(x_i^T\beta + \beta_0)\}$. Indeed we can argue that we have $=$ at solution

Therefore plugging in for optimal ξ gives the hinge form of SVMs:

$$\min_{\beta,\beta_0} \ \frac{1}{2}\|\beta\|_2^2 + C\sum_{i=1}^n \big[1 - y_i(x_i^T\beta + \beta_0)\big]_+$$

where $a_+ = \max\{0, a\}$ is called the hinge function

Transformations and change of variables

If $h : \mathbb{R} \to \mathbb{R}$ is a monotone increasing transformation, then

$$\begin{aligned} &\min \; f(x) \;\text{ subject to }\; x \in C \\ \iff &\min \; h(f(x)) \;\text{ subject to }\; x \in C \end{aligned}$$

Similarly, inequality or equality constraints can be transformed and yield equivalent optimization problems. Can use this to reveal the "hidden convexity" of a problem

If $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, and its image covers feasible set C, then we can change variables in an optimization problem:

$$\begin{aligned} &\min_x \; f(x) \;\text{ subject to }\; x \in C \\ \iff &\min_y \; f(\phi(y)) \;\text{ subject to }\; \phi(y) \in C \end{aligned}$$

Example: geometric programming

A monomial is a function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ of the form

$$f(x) = \gamma x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

for $\gamma > 0$, $a_1, \ldots a_n \in \mathbb{R}$. A posynomial is a sum of monomials,

$$f(x) = \sum_{k=1}^{p} \gamma_k x_1^{a_{k1}} x_2^{a_{k2}} \cdots x_n^{a_{kn}}$$

A geometric program of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 1, \ i = 1, \ldots m \\ & h_j(x) = 1, \ j = 1, \ldots r \end{aligned}$$

where f, g_i, $i = 1, \ldots m$ are posynomials and h_j, $j = 1, \ldots r$ are monomials. This is nonconvex

Let's prove that a geometric program is equivalent to a convex one. Given $f(x) = \gamma x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, let $y_i = \log x_i$ and rewrite this as

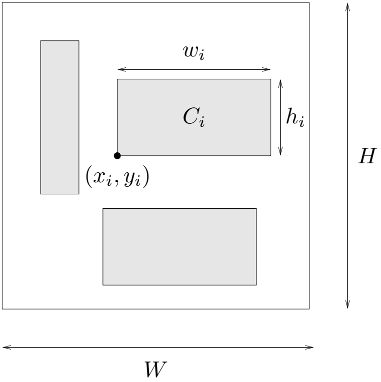
$$\gamma(e^{y_1})^{a_1}(e^{y_2})^{a_2}\cdots(e^{y_n})^{a_n} = e^{a^T y + b}$$

for $b = \log \gamma$. Also, a posynomial can be written as $\sum_{k=1}^{p} e^{a_k^T y + b_k}$. With this variable substitution, and after taking logs, a geometric program is equivalent to

$$\begin{aligned} \min_y \quad & \log\left(\sum_{k=1}^{p_0} e^{a_{0k}^T y + b_{0k}}\right) \\ \text{subject to} \quad & \log\left(\sum_{k=1}^{p_i} e^{a_{ik}^T y + b_{ik}}\right) \le 0,\ i = 1, \ldots m \\ & c_j^T y + d_j = 0,\ j = 1, \ldots r \end{aligned}$$

This is convex, recalling the convexity of soft max functions

Many interesting problems are geometric programs, e.g., floor planning:



See Boyd et al. (2007), "A tutorial on geometric programming", and also Chapter 8.8 of B & V book

Example floor planning program:

$$\begin{aligned} \min_{\substack{W,H,\\x,y,w,h}} \quad & WH \\ \text{subject to} \quad & 0 \le x_i \le W,\ i = 1, \ldots n \\ & 0 \le y_i \le H,\ i = 1, \ldots n \\ & x_i + w_i \le x_j,\ (i,j) \in \mathcal{L} \\ & y_i + h_i \le y_j,\ (i,j) \in \mathcal{B} \\ & w_i h_i = C_i,\ i = 1, \ldots n. \end{aligned}$$

Check: why is this a geometric program?

Eliminating equality constraints

Important special case of change of variables: eliminating equality constraints. Given the problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0,\ i = 1, \dots m \\ & Ax = b \end{aligned}$$

we can always express any feasible point as $x = My + x_0$, where $Ax_0 = b$ and $\mathrm{col}(M) = \mathrm{null}(A)$. Hence the above is equivalent to

$$\begin{aligned} \min_y \quad & f(My + x_0) \\ \text{subject to} \quad & g_i(My + x_0) \le 0,\ i = 1, \dots m \end{aligned}$$

Note: this is fully general but not always a good idea

Introducing slack variables

Opposite to eliminating equality constraints: introducing slack variables. Given the problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{subject to} \quad & g_i(x) \le 0,\ i = 1, \ldots m \\
& Ax = b
\end{aligned}
$$

we can transform the inequality constraints via

$$
\begin{aligned}
\min \quad & f(x) \\
\text{subject to} \quad & s_i \ge 0,\ i = 1, \ldots m \\
& g_i(x) + s_i = 0,\ i = 1, \ldots m \\
& Ax = b
\end{aligned}
$$

Note: this is no longer convex unless each $g_i(x) = c_i^T x + d_i$, an affine function

Relaxing nonaffine equality constraints

Given an optimization problem

$$\min \ f(x) \ \text{subject to} \ x \in C$$

we can always take an enlarged constraint set $\tilde{C} \supseteq C$ and consider

$$\min \ f(x) \ \text{subject to} \ x \in \tilde{C}$$

This is called a relaxation and its optimal value is always smaller or equal to that of the original problem

Important special case: relaxing nonaffine equality constraints, i.e.,

$$h_j(x) = 0, \ j = 1, \ldots r$$

where h_j, $j = 1, \ldots r$ are convex but nonaffine, are replaced with

$$h_j(x) \leq 0, \ j = 1, \ldots r$$

Example: maximum utility problem

The maximum utility problem models investment/consumption:

$$\begin{array}{ll} \max_{x,b} & \sum_{t=1}^{T} \alpha_t u(x_t) \\ \text{subject to} & b_{t+1} = b_t + f(b_t) - x_t,\ t = 1, \dots T \\ & 0 \le x_t \le b_t,\ t = 1, \dots T \end{array}$$

Here b_t is the budget and x_t is the amount consumed at time t; f is an investment return function, u utility function, both concave and increasing

Is this a convex problem? What if we replace equality constraints with inequalities:

$$b_{t+1} \le b_t + f(b_t) - x_t,\ t = 1, \dots T?$$

Example: principal components analysis

Given $X \in \mathbb{R}^{n \times p}$, consider the low rank approximation problem:

$$\min_{R \in \mathbb{R}^{n \times p}} \|X - R\|_F^2 \ \text{ subject to } \ \text{rank}(R) = k$$

Here $\|A\|_F^2 = \sum_{i=1}^n \sum_{j=1}^p A_{ij}^2$, the entrywise squared ℓ_2 norm, and $\text{rank}(A)$ denotes the rank of A

Well-known solution is given by singular value decomposition or SVD: if $X = UDV^T$, then solution is

$$R = U_k D_k V_k^T$$

where U_k, V_k are the first k columns of U, V and D_k is the first k diagonal elements of D. I.e., R is reconstruction of X from its first k principal components

This problem is not convex. Why?

We can recast the principal components analysis or PCA problem in a convex form. First rewrite as

$$\min_{Z \in \mathbb{S}^p} \|X - XZ\|_F^2 \;\; \text{subject to} \;\; \operatorname{rank}(Z) = k, \;\; Z \text{ is a projection}$$
$$\iff \max_{Z \in \mathbb{S}^p} \operatorname{tr}(X^T X Z) \;\; \text{subject to} \;\; \operatorname{rank}(Z) = k, \;\; Z \text{ is a projection}$$

Hence constraint set is the nonconvex set

$$C = \Big\{ Z \in \mathbb{S}^p : \lambda_i(Z) \in \{0,1\}, \; i = 1, \ldots p, \; \operatorname{tr}(Z) = k \Big\}$$

where $\lambda_i(Z)$, $i = 1, \ldots n$ are the eigenvalues of Z

Solution here is

$$Z = V_k V_k^T$$

where V_k gives first k columns of V

Now consider relaxing constraint set to $\mathcal{F} = \mathrm{conv}(C)$, its convex hull. But

$$\begin{aligned}\mathcal{F} &= \{Z \in \mathbb{S}^p : \lambda_i(Z) \in [0,1],\ i = 1, \dots p,\ \mathrm{tr}(Z) = k\} \\ &= \{Z \in \mathbb{S}^p : 0 \preceq Z \preceq I,\ \mathrm{tr}(Z) = k\}\end{aligned}$$

This is exactly the fantope of order k

Hence the fantope projection problem

$$\min_{Z \in \mathbb{S}^p} \ \mathrm{tr}(X^T X Z) \ \text{ subject to } \ Z \in \mathcal{F}$$

is convex. Remarkably, this is equivalent to the nonconvex PCA problem (admits the same solution)!

(Famous result: Fan (1949), "On a theorem of Weyl conerning eigenvalues of linear transformations")

References and further reading

- S. Boyd and L. Vandenberghe (2004), "Convex optimization", Chapter 4
- O. Guler (2010), "Foundations of optimization", Chapter 4