Duality gap:

observe $f(x^*) \geq g(u,v)$ any dual feas u,v

hence $f(x) - f(x^*) \leq f(x) - g(u,v)$

likewise

$$g(u^*,v^*) - g(u,v) \leq f(x) - g(u,v)$$

From stationarity

x^* minimizes $L(x, u^*, v^*)$

$$L(x,v) = \sum_{i=1}^{n} \left(f_i(x_i) + \bar{a_i x_i v} \right) + bv$$

$$\min_x L(x,v) = \sum_{i=1}^{n} \underbrace{\min_{x_i} \left(f_i(x_i) - a_i v x_i \right)}_{-f_i^*(a_i v)} + bv$$

$$\begin{array}{c} \shortparallel \\ g(v) \end{array}$$

$$\|x\|_* = \max_{\|z\| \leq 1} z^T x$$

$$\geq \left(\frac{y}{\|y\|} \right)^T x$$

hence $|y^T x| \leq \|y\| \|x\|_*$

$$f^*(y) = \max_x \; x^T y - f(x)$$

$$\geq x^T y - f(x)$$

quad $\quad f(x) = \frac{1}{2} x^T Q x, \quad Q \succ 0$

$$f^*(y) = \max_x \; y^T x - \frac{1}{2} x^T Q x$$

$$= -\min_x \; \underbrace{\frac{1}{2} x^T Q x - x^T y}$$

$$0 = \nabla(\;) = Qx - y$$

$$\iff x = Q^{-1} y$$

$$f^*(y) = \frac{1}{2} y^T Q^{-1} y$$

Fenchel's ineq. $\quad x^T y \leq \frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y$

ind. $\quad f(x) = I_C(x)$

$$f^*(y) = \max_x \; y^T x - I_C(x)$$

$$= \max_{x \in C} \; y^T x$$

norms.

- use $f^{**} = f$
- hence conj. of support fn is indicator
- apply to norms

Lasso:

$$L(\beta, z, u) = \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$$

$$g(u) = \min_{\beta, z} L(\beta, z, u)$$

$$= \underbrace{\min_{\beta} \lambda\|\beta\|_1 - (X^T u)^T\beta}_{= -\lambda \max_{\beta}\left(\left(\frac{(X^T u)}{\lambda}\right)^T \beta - \|\beta\|_1\right) = -\lambda \cdot I\left\{\left\|\frac{X^T u}{\lambda}\right\|_\infty \le 1\right\}} + \underbrace{\min_{z} \frac{1}{2}\|y - z\|_2^2 + u^T z}_{= \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2}$$

ie. dual prob is

$$\max_{u} \; -\frac{1}{2}\|y - u\|_2^2$$

$$\text{s.t.} \quad \|X^T u\|_\infty \le \lambda$$

β^*, z^* must minimize $L(\beta, z, u^*)$

$$0 = \nabla_z L(\beta, z, u^*) \iff z^* = y - u^*$$

ie. $X\beta^* = y - u^*$