Duality revisited

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Last time: barrier method

Main idea: approximate the problem

$$\begin{array}{rl} \min\limits_x & f(x) + I_C(x) \\ \text{subject to} & Ax = b \end{array}$$

with the barrier problem

$$\begin{array}{rl} \min\limits_x & f(x) + \frac{1}{t}\phi(x) \\ \text{subject to} & Ax = b \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min\limits_x & tf(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

where $t > 0$ and ϕ is a barrier function for C.

Logarithmic barrier

Common case:

$$C = \{x : h_i(x) \le 0,\ i = 1, \dots, m\}.$$

Logarithmic barrier

$$\phi(x) = -\sum_{i=1}^{m} \log(-h_i(x))$$

Nice fact: when f, h_i are smooth and KKT hold for both original and barrier problem, the solution $x^\star(t)$ to barrier problem satisfies

$$f(x^\star(t)) - f^\star \le m/t.$$

Strict feasibility

Important detail: throughout the algorithm, line-search should be performed so that the iterates satisfy

$$h_i(x) < 0, \quad i = 1, \dots m.$$

To find an initial x for the barrier problem solve

$$\begin{array}{ll} \min\limits_{x,s} & s \\ \text{subject to} & h_i(x) \le s, \quad i = 1, \dots m \\ & Ax = b. \end{array}$$

Stop early: as soon as we find a feasible solution with $s < 0$.

If the above minimum is positive, then original problem is infeasible.

Outline

Today

- Lagrangian duality revisited
- Optimality conditions
- Connection with barrier problems
- Fenchel duality

Lagrangian duality revisited

Consider the primal problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & h_i(x) \le 0,\ i = 1, \ldots m \\ & \ell_j(x) = 0,\ j = 1, \ldots r \end{aligned}$$

Lagrangian

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x)$$

We can rewrite the primal problem as

$$\min_x \ \max_{\substack{u,v \\ u \ge 0}} \ L(x, u, v)$$

Dual problem

$$\max_{\substack{u,v \\ u \ge 0}} \ \min_x \ L(x, u, v)$$

Weak and strong duality

Theorem (weak duality)

Let p and d denote the optimal values of the above primal and dual problems respectively. Then $p \geq d$.

Theorem (strong duality)

Assume $f, h_1, \ldots, h_p$ are convex with domain D and $h_{p+1}, \ldots, h_m, \ell_1, \ldots, \ell_r$ are affine.
If there exists $\hat{x} \in$ relint(D) such that

$$h_i(\hat{x}) < 0,\ i = 1, \ldots, p;\ h_i(\hat{x}) \leq 0,\ i = p+1, \ldots, m$$

and

$$\ell_j(\hat{x}) = 0,\ j = 1, \ldots, r$$

then $p = d$ and the dual optimum is attained if finite.

Example: linear programming

Primal problem (in standard form)

$$\begin{aligned} \min_{x} \quad & c^{\mathsf{T}} x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \max_{y,s} \quad & b^{\mathsf{T}} y \\ \text{subject to} \quad & A^{\mathsf{T}} y + s = c \\ & s \geq 0 \end{aligned}$$

Example: convex quadratic programming

Primal problem (in standard form)

$$\begin{array}{rl} \min\limits_{x} & \frac{1}{2}x^\mathsf{T}Qx + c^\mathsf{T}x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where Q symmetric and positive semidefinite.

Dual problem

$$\begin{array}{rl} \max\limits_{u,y,s} & b^\mathsf{T}y - \frac{1}{2}u^\mathsf{T}Qu \\ \text{subject to} & A^\mathsf{T}y + s - c = Qu \\ & s \geq 0 \end{array}$$

Example: barrier problem for linear programming

Primal problem

$$\begin{aligned} \min_{x} \quad & c^{\mathsf{T}} x - \tau \sum_{i=1}^{n} \log(x_i) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where $\tau > 0$.

Dual problem

$$\begin{aligned} \max_{y,s} \quad & b^{\mathsf{T}} y + \tau \sum_{i=1}^{n} \log(s_i) + n(\tau - \tau \log \tau) \\ \text{subject to} \quad & A^{\mathsf{T}} y + s = c \end{aligned}$$

Optimality conditions

Consider the problem

$$\begin{array}{rl} \min\limits_{x} & f(x) \\ \text{subject to} & Ax = b \\ & h(x) \leq 0 \end{array}$$

Here $h(x) \leq 0$ is shorthand for $h_i(x) \leq 0, \;\; i = 1, \ldots m$.

Assume $f, h_1, \ldots, h_m$ are convex and differentiable. Assume also that strong duality holds.

Then $x^\star$ and $(u^\star, v^\star)$ are respectively primal and dual optimal solutions if and only if $(x^\star, u^\star, v^\star)$ solves the KKT conditions

$$\begin{aligned} \nabla f(x) + A^\mathsf{T} v + \nabla h(x) u &= 0 \\ Ax &= b \\ U h(x) &= 0 \\ u, -h(x) &\geq 0. \end{aligned}$$

Here $U = \mathrm{Diag}(u)$, $\nabla h(x) = \begin{bmatrix} \nabla h_1(x) & \cdots & \nabla h_m(x) \end{bmatrix}$

Central path equations

Barrier problem

$$\begin{aligned} \min_x \quad & f(x) + \tau\phi(x) \\ & Ax = b \end{aligned}$$

where

$$\phi(x) = -\sum_{i=1}^{m} \log(-h_i(x)).$$

Optimality conditions for barrier problem (and its dual)

$$\begin{aligned} \nabla f(x) + A^{\mathsf{T}} v + \nabla h(x) u &= 0 \\ Ax &= b \\ Uh(x) &= -\tau 1 \\ u, -h(x) &> 0. \end{aligned}$$

Special case: linear programming

Primal and dual problems

$$\begin{array}{rl} \min\limits_{x} & c^{\mathsf{T}} x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{rl} \max\limits_{y,s} & b^{\mathsf{T}} y \\ \text{subject to} & A^{\mathsf{T}} y + s = c \\ & s \geq 0 \end{array}$$

Optimality conditions for both

$$\begin{aligned} A^{\mathsf{T}} y + s &= c \\ Ax &= b \\ XS1 &= 0 \\ x, s &\geq 0. \end{aligned}$$

Here $X = \mathsf{Diag}(x),\ S = \mathsf{Diag}(s)$.

Algorithms for linear programming

Recall the optimality conditions for linear programming

$$\begin{aligned} A^\mathsf{T} y + s &= c \\ Ax &= b \\ XS1 &= 0 \\ x, s &\geq 0. \end{aligned}$$

Two main classes of algorithms

- Simplex: maintain first three and aim for fourth one
- Interior-point methods: maintain fourth (and maybe first and second) and aim for third one.

Central path for linear programming

Primal and dual barrier problems

$$\begin{array}{rl} \min\limits_{x} & c^{\mathsf{T}} x - \tau \sum_{i=1}^{n} \log(x_i) \\ \text{subject to} & Ax = b \end{array} \qquad \begin{array}{rl} \max\limits_{y,s} & b^{\mathsf{T}} y + \tau \sum_{i=1}^{n} \log(s_i) \\ \text{subject to} & A^{\mathsf{T}} y + s = c \end{array}$$

Optimality conditions for both

$$\begin{aligned} A^{\mathsf{T}} y + s &= c \\ Ax &= b \\ XS1 &= \tau 1 \\ x, s &> 0. \end{aligned}$$

Fenchel duality

Consider the primal problem

$$\min_x \; f(x) + g(Ax)$$

Rewrite it as

$$\begin{array}{rl} \min\limits_x & f(x) + g(z) \\ \text{subject to} & Ax = z. \end{array}$$

Dual problem

$$\max_v \; -f^*(A^\mathsf{T} v) - g^*(-v).$$

This special type of duality is called Fenchel duality.

Nice fact: if f, g are convex and closed then the dual of the dual is the primal.

Example: conic programming

Primal problem (in standard form)

$$
\begin{array}{rl}
\min\limits_{x} & c^{\mathsf{T}} x \\
\text{subject to} & Ax = b \\
& x \in K
\end{array}
$$

where K is a closed convex cone.

Dual problem

$$
\begin{array}{rl}
\max\limits_{y,s} & b^{\mathsf{T}} y \\
\text{subject to} & A^{\mathsf{T}} y + s = c \\
& s \in K^*
\end{array}
$$

Strong duality holds if one of the problems is strictly feasible.

If both primal and dual are strictly feasible, then strong duality holds and both primal and dual optima are attained.

Example: semidefinite programming

Primal

$$\begin{array}{ll} \min\limits_{X} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \ i = 1, \ldots, m \\ & X \succeq 0. \end{array}$$

Dual

$$\begin{array}{ll} \max\limits_{y} & b^{\mathsf{T}} y \\ \text{subject to} & \sum\limits_{i=1}^{m} y_i A_i + S = C \\ & S \succeq 0. \end{array}$$

Recall trace inner product in $\mathbb{S}^n$

$$X \bullet S = \mathsf{trace}(XS).$$

Strong duality does not always hold

Examples

$$\begin{aligned} \min \quad & 2x_{12} \\ & \begin{bmatrix} 0 & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & x_{11} \\ & \begin{bmatrix} x_{11} & 1 \\ 1 & x_{22} \end{bmatrix} \succeq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & ax_{22} \\ & \begin{bmatrix} 0 & x_{12} & 1 - x_{22} \\ x_{12} & x_{22} & x_{23} \\ 1 - x_{22} & x_{23} & x_{33} \end{bmatrix} \succeq 0, \quad \text{for} \;\; a > 0. \end{aligned}$$

Example: barrier problem for semidefinite programming

Primal

$$\begin{array}{ll} \min\limits_{X} & C \bullet X - \tau \log(\det(X)) \\ \text{subject to} & A_i \bullet X = b_i,\ i = 1, \ldots, m \end{array}$$

Dual

$$\begin{array}{ll} \max\limits_{y,S} & b^{\mathsf{T}} y + \tau \log(\det(S)) + n(\tau - \tau \log \tau) \\ \text{subject to} & \sum_{i=1}^{m} y_i A_i + S = C \end{array}$$

Optimality conditions for semidefinite programming

Primal and dual problems

$$\begin{array}{rl} \min\limits_{X} & C \bullet X \\ \text{subject to} & \mathcal{A}(X) = b \\ & X \succeq 0 \end{array} \qquad \begin{array}{rl} \max\limits_{y,S} & b^\mathsf{T} y \\ \text{subject to} & \mathcal{A}^*(y) + S = C \\ & C \succeq 0 \end{array}$$

Here $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ linear map.

Assume also that strong duality holds. Then $X^\star$ and $(y^\star, S^\star)$ are respectively primal and dual optimal solutions if and only if $(X^\star, y^\star, S^\star)$ solves

$$\begin{aligned} \mathcal{A}^*(y) + S &= C \\ \mathcal{A}(X) &= b \\ XS &= 0 \\ X, S &\succeq 0. \end{aligned}$$

Central path for semidefinite programming

Primal barrier problem

$$\begin{array}{rl} \min\limits_{X} & C \bullet X - \tau \log(\det(X)) \\ \text{subject to} & \mathcal{A}(X) = b \end{array}$$

Dual barrier problem

$$\begin{array}{rl} \max\limits_{y,S} & b^{\mathsf{T}} y + \tau \log(\det(S)) \\ \text{subject to} & \mathcal{A}^*(y) + S = C \end{array}$$

Optimality conditions for both

$$\begin{aligned} \mathcal{A}^*(y) + S &= C \\ \mathcal{A}(X) &= b \\ XS &= \tau I \\ X, S &\succ 0. \end{aligned}$$

References

- O. Guler (2010), “Foundations of Optimization”, Chapter 11
- J. Renegar (2001), “A mathematical view of interior-point methods in convex optimization,” Chapters 2 and 3.
- S. Wright (1997), “Primal-dual interior-point methods”, Chapters 5 and 6.