1

$$\min_{x,z} \quad f(x) + g(z)$$
$$\text{s.t.} \quad x = z$$

ADMM

x: $$x^+ = \underset{x}{\operatorname{argmin}} \ f(x) + \frac{\rho}{2}\|x - z + w\|_2^2$$
$$= \operatorname{argmin} \ \frac{1}{2 \cdot 1/\rho}\|z - w - x\|_2^2 + f(x)$$
$$= \operatorname{prox}_{f, 1/\rho}(z - w)$$

z: $$z^+ = \operatorname{prox}_{g, 1/\rho}(x^+ + w)$$

w: $$w^+ = w + (x^+ - z^+)$$

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lasso

$$\underbrace{\tfrac{1}{2}\|y - X\beta\|_2^2}_{f(\beta)} + \underbrace{\lambda\|\alpha\|_1}_{g(\alpha)}$$

ADMM

β: $$\beta^+ = \operatorname{argmin} \ f(\beta) + \frac{\rho}{2}\|\beta - \alpha + w\|_2^2$$
$$= \operatorname{argmin} \ \frac{1}{2}\|y - X\beta\|_2^2 + \frac{\rho}{2}\|\beta - \alpha + w\|_2^2$$
$$= (X^T X + \rho I)^{-1}(X^T y + \rho(\alpha - w))$$

α: $$\alpha^+ = \operatorname{argmin} \ g(\alpha) + \frac{\rho}{2}\|\beta^+ - \alpha + w\|_2^2$$
$$= \operatorname{argmin} \ \lambda\|\alpha\|_1 + \frac{\rho}{2}\|\beta^+ + w - \alpha\|_2^2$$
$$= S_{\lambda/\rho}(\beta^+ + w)$$

w: $$w^+ = w + \beta^+ - z^+$$

②

$$S = X^T X \qquad X \quad n \times p$$

$$\min_P \; \|X - XP\|_F^2 \quad \text{st. } P \text{ projection matrix.}$$
$$\text{rank}(P) = k$$

$$\Longleftrightarrow \max_Y \; \mathrm{tr}(SY) \quad \text{st. } Y \in \mathcal{F}_k = \{ Y : 0 \preceq Y \preceq I,\ \mathrm{tr}(Y) = k \}$$

$$\max \; \mathrm{tr}(SY) - \lambda \|Y\|_1$$
$$\text{st} \quad Y \in \mathcal{F}_k$$

$$\Longleftrightarrow \min \; -\mathrm{tr}(SY) + I_{\mathcal{F}_k}(Y) + \lambda \|Z\|_1$$

$$Y: \; Y^+ = \operatorname*{argmin}_Y \; -\mathrm{tr}(SY) + I_{\mathcal{F}_k}(Y) + \frac{\rho}{2} \|Y - Z + W\|_F^2$$

$$= \operatorname*{argmin}_{Y \in \mathcal{F}_k} \; \frac{1}{2} \|Y - Z + W \cancel{+} - \frac{S}{\rho}\|_F^2$$

$$= P_{\mathcal{F}_k}\left(Z - W + S/\rho\right)$$

$$Z: \; Z^+ = \operatorname*{argmin}_Z \; \lambda \|Z\|_1 + \frac{\rho}{2} \|Y^+ - Z + W\|_F^2$$

$$= S_{\lambda/\rho}(Y^+ + W)$$

$$W: \quad W^+ = W + Y^+ - Z^+.$$

③

$$\min_{x} \sum_{i=1}^{B} f_i(x)$$

$$\Longleftrightarrow \quad \min_{x_1, \dots, x_B} \sum_{i=1}^{B} f_i(x_i) \qquad \begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 \\ &\vdots \\ x_{B-1} &= x_B \end{aligned}$$

$$\Longleftrightarrow \quad \min_{x, x_1, \dots, x_B} \sum_{i=1}^{B} f_i(x_i) \qquad \begin{aligned} x_1 &= x \\ x_2 &= x \\ &\vdots \\ x_B &= x \end{aligned}$$

ADMM

parallel

x_1: $x_1^+ = \underset{x_1}{\operatorname{argmin}} \ f_1(x_1) + \frac{\rho}{2} \| x_1 - x + w_1 \|_2^2$

x_2: $x_2^+ = \underset{x_2}{\operatorname{argmin}} \ f_2(x_2) + \frac{\rho}{2} \| x_2 - x + w_2 \|_2^2$

$\vdots$

x_B: $x_B^+ = \underset{x_B}{\operatorname{argmin}} \ f_B(x_B) + \frac{\rho}{2} \| x_B - x + w_B \|_2^2$

x^+ = simple average

w^+ = as usual