Observe:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax-b \end{bmatrix}$$

$\iff$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w-y \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T y \\ Ax-b \end{bmatrix}$$

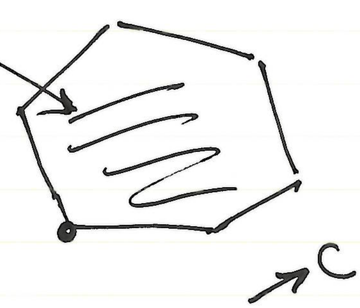
Special case: when $Ax = b$, get

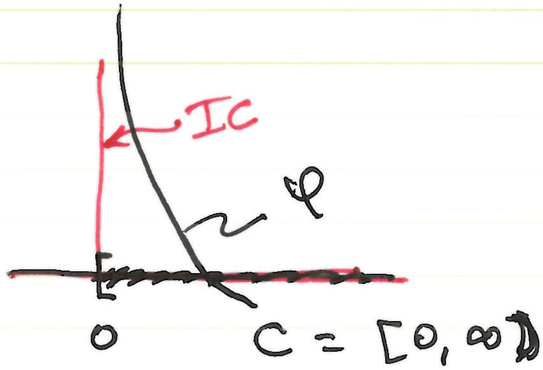
$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix}$$

Special case: LP

$$\min_x \; c^T x$$
$$Dx \le e$$

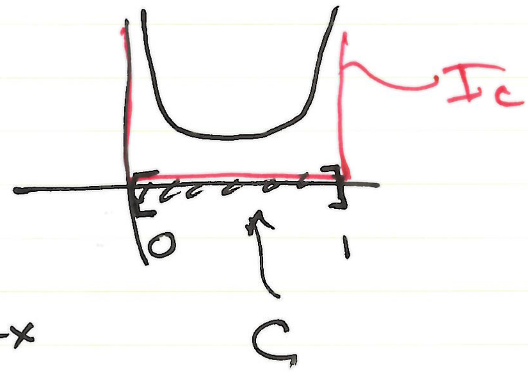
$\{x : Dx \le e\}$





barrier: $-\log(x) = \varphi(x)$

More interesting pic



barrier: $-\log(x) - \log(1-x$

<u>Example</u>

$$\varphi(x) = -\sum_{i=1}^{n} \log x_i \iff h_i(x) = -x_i$$

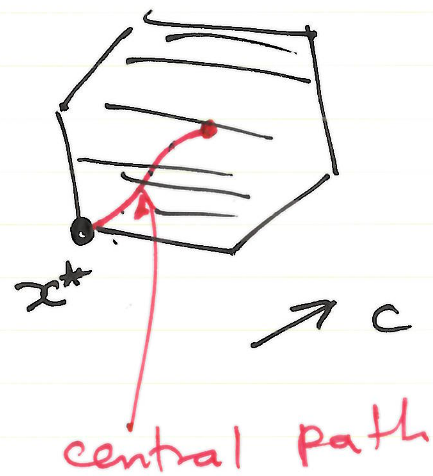
$$\nabla \varphi(x) = -\begin{bmatrix} 1/x_1 \\ \vdots \\ 1/x_n \end{bmatrix} = -X^{-1}\mathbb{1}$$

$$X = \mathrm{Diag}(x)$$

$$\nabla^2 \varphi(x) = \begin{bmatrix} 1/x_i^2 & & \\ & \ddots & \\ & & 1/x_n^2 \end{bmatrix} = X^{2}$$

LP $\min c^T x$

$Dx \leq e$



Observe: KKT condns for barrier problem

Put $u_i(t) := -\dfrac{1}{t\, h_i(x^*(t))}$, $v := \frac{1}{t} w$

Rewrite as follows

$$\begin{cases} \nabla f(x^*(t)) + \sum_{i=1}^{m} u_i(t) \nabla h_i(x^*(t)) + A^T v = 0 \\ A x^*(t) = b, \quad u_i(t) h_i(x^*(t)) = -\frac{1}{t} \\ \qquad h_i(x^*(t)) < 0 \\ \qquad u_i(t) > 0 \end{cases}$$

Duality gap derivation

$$f(x^*(t)) - f(x^*) + \sum_{i=1}^{m} u_i(t)\big(h_i(x^*(t)) - h_i(x^*)\big) \qquad 0$$

$$\leq \Big\langle \underbrace{\nabla f(x^*(t)) + \sum_{i=1}^{m} u_i(t) \nabla h_i(x^*(t))}_{-A^T v},\ \cancel{x^*(t) - x^*} \Big\rangle$$

$$\Rightarrow \quad f(x^*(t)) - f(x^*) \leq \frac{m}{t} + \sum_{i=1}^{m} u_i(t) \underbrace{h_i(x^*(t))}_{\leq 0}$$

$$\leq \frac{m}{t}.$$

An example of a SCB:

$$\phi(x) = - \sum \log x_i$$

$$\lambda(x)^2 = n$$