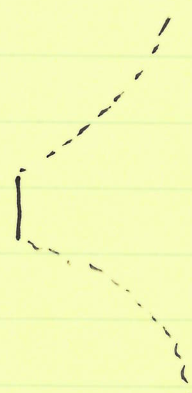
1

$$f(u) = \|y-u\|_2^2$$
$$f(X\beta) = \|y - X\beta\|_2^2$$

$$g(u,v) \le f^*$$
$$-g(u,v) \ge -f^*$$
$$\underbrace{f(x) - g(u,v)}_{\text{duality gap}} \ge f(x) - f^*$$
$$\ge g^* - g(u,v)$$



$$\min_x \sum f_i(x_i) - v a^T x$$
$$= \min_x \sum f_i(x_i) - v \sum a_i x_i$$
$$= \sum \underbrace{\left(\min_{x_i} f_i(x_i) - v a_i x_i \right)}_{-f_i^*(a_i v)}$$

$$\|x\|_* = \max_{\|z\| \le 1} |z^T x|$$

arbitrary y, $z = \frac{y}{\|y\|}$. then $\|z\| = 1$.

also $|z^T x| \le \|x\|_*$

so $|y^T x| \le \|y\| \|x\|_*$

if $\|u\|_* \le 1$, then

$$\max_y \left(y^T u - \|y\| \right) = 0$$

$$y^T u \le \|y\| \, \|u\|_* \le \|y\|$$

if $\|u\|_* > 1$, then

$$\max_y \left(y^T u - \|y\| \right) = \infty$$

can always find z with $\|z\| = 1$,
s.t. $\quad z^T u = \|u\|_* \quad \left(\operatorname*{argmax}_{\|z\| \le 1} z^T u \right)$

take $y = t \cdot z$, $t > 0$

$$y^T u = t \cdot \|u\|_*$$

for this y, $\quad y^T u - \|y\| = t \cdot \|u\|_* - t$

$$\to \infty \quad \text{as } t \to \infty$$

so therefore

$$g(u) = \min_y L(y, u)$$

$$= x^T u - I\left(\|u\|_* \le 1 \right)$$

dual problem

$$\max_u g(u) \iff \max_{\|u\|_* \le 1} x^T u$$

$$f(x) + f^*(y) \ge x^T y$$

$$\iff f^*(y) \ge x^T y - f(x)$$

$$\max_z z^T y - f(z)$$

③

$$f(x) = \frac{1}{2} x^T Q x, \qquad Q \succ 0$$

$$f^*(y) = \max_x \left(y^T x - \frac{1}{2} x^T Q x \right)$$

$$= - \min_x \left(\frac{1}{2} x^T Q x - y^T x \right)$$

$ax^2 + bx + c$

$\frac{-b}{2a}$

$$x^* = Q^{-1} y$$

$$= - \frac{1}{2} y^T Q^{-1} Q Q^{-1} y + y^T Q^{-1} y$$

$$= \frac{1}{2} y^T Q^{-1} y$$

Fenchel's inequality: for any x, y

$$\frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y \geq x^T y$$

$$f(\beta) = \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \|\beta\|_1$$

$$L(\beta) = f(\beta)$$

$$\min_\beta L(\beta) = f^*$$

$$\min_{\beta, z} \ \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$$

$$= \left(\min_z \ \frac{1}{2}\|y - z\|_2^2 + u^T z \right) + \left(\min_\beta \ \lambda\|\beta\|_1 - u^T X \beta \right)$$

$$z^* = y - u$$

$$= \frac{1}{2}\|u\|_2^2 + u^T(y - u)$$

$$= -\frac{1}{2}\|y - u\|_2^2 + \frac{1}{2}\|y\|_2^2$$

$\cancel{z = y + u}$

$\cancel{z = y} u$

$$= -\lambda \cdot \max_\beta \left(\frac{u^T X}{\lambda} \beta - \|\beta\|_1 \right)$$

$$= -\lambda \cdot I\left(\left\| \frac{X^T u}{\lambda} \right\|_\infty \leq 1 \right)$$

$$= -\lambda \cdot I\left(\|X^T u\|_\infty \leq \lambda \right)$$

strong duality $\Rightarrow$ (β^*, z^*) must minimize $L(\beta, z, u^*)$ over all β, z