①

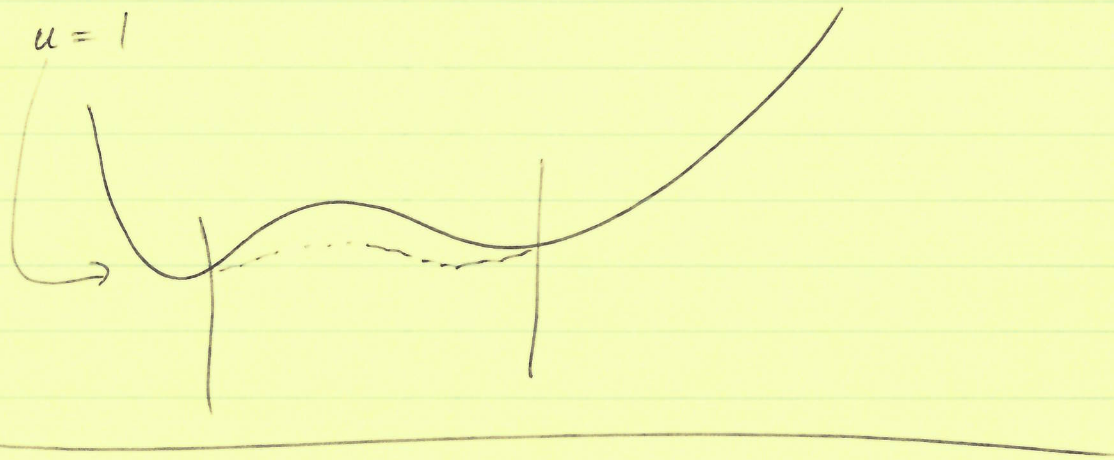
$$L(x,u,v) = (c + A^T u + G^T v)^T x - b^T u - h^T v$$

$$\min_x \quad f(x)$$

$$\text{s.t.} \quad h(x) \le 0$$

$$L(x,u) = f(x) + u h(x)$$

$u = 1$



$$L(x,u,v) = \tfrac{1}{2} x^T Q x + c^T x + \sum u_i(-x_i) + \sum v_j (Ax-b)_j$$

$$= \tfrac{1}{2} x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

$$g(u,v) = \min_x L(x,u,v) = \tfrac{1}{2} x^T Q x + (c - u + A^T v)^T x - b^T v.$$

$$x^* = -Q^{-1}(c - u + A^T v)$$

$$\min_x L(x,u,v) = L(x^*, u, v)$$

$$= \tfrac{1}{2}(c - u + A^T v)^T Q^{-1} (c - u + A^T v)$$

$$- (c - u + A^T v)^T Q^{-1} (c - u + A^T v)$$

$$- b^T v$$

$$= -\tfrac{1}{2}(c - u + A^T v)^T Q^{-1}(c - u + A^T v) - b^T v$$

$ax^2 + bx + c$

$x^* = -\frac{b}{2a}$

$\frac{1}{2} x^T Q x + b^T x$

$x^* = -Q^{-1} b.$

②

$$L(x,u,v) = \frac{1}{2}x^TQx + (c-u+A^Tv)^Tx - b^Tv$$

$$Qx = -(c-u+A^Tv)$$

(i) $c-u+A^Tv \in \mathrm{col}(Q) \iff c-u+A^Tv \perp \mathrm{null}(Q)$

$x^* = -Q^+(c-u+A^Tv)$

$$Q = UDU^T = U\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0\end{pmatrix}U^T$$

$$Q^+ = U\begin{pmatrix} 1/d_1 & & & & \\ & \ddots & & & \\ & & 1/d_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0\end{pmatrix}U^T$$

definition of pseudoinverse

(ii) $c-u+A^Tv \not\perp \mathrm{null}(Q)$

$c-u+A^Tv = z_1+z_2$

$z_1 \in \mathrm{col}(Q)$, $z_2 \in \mathrm{null}(Q)$

$z_2 \neq 0$

$$\min_x L(x,u,v) = -\infty.$$

$$g(u,v) = \begin{cases} -\frac{1}{2}(c-u+A^Tv)^TQ^+(c-u+A^Tv) - b^Tv & \text{in case (i)} \\ -\infty & \text{in case (ii)} \end{cases}$$

case (i): $c-u+A^Tv \perp \mathrm{null}(Q)$

$\iff P(c-u+A^Tv) = 0$

where P = projection matrix onto $\mathrm{null}(Q)$

$$\min_{x=(x_1,x_2)} f(x) = \frac{1}{2}x^TQx + b^Tx$$

s.t. $x \geq 0$

$$g(u) = \min_x \frac{1}{2}x^TQx + b^Tx - u^Tx = \text{quad. in } u=(u_1,u_2)$$

QP dual:

$$\max_{u,v} \quad -\frac{1}{2}(c-u+A^Tv)^T Q^+ (c-u+A^Tv)$$
$$\text{s.t.} \quad P(c-u+A^Tv) = 0, \quad u \geq 0.$$

$$L(x,u) = x^4 - 50x^2 + 100x - ux - 4.5u$$

$\min_x L(x,u)$: $4x^3 - 100x + 100 - u = 0$

solved for all roots

... $\Rightarrow$ dual function

Slater's condition for LPs: get strong duality when LP is feasible

min	max
primal LP	dual LP
$f^\star$	$g^\star$

primal feasible: $f^\star = g^\star$

dual feasible: $g^\star = f^\star$

both infeasible: $\underset{-\infty}{g^\star} \neq \underset{\infty}{f^\star}$