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$$\min_{x,y} \quad x + 3y$$

s.t. $\quad x + y \geq 2$

$x \geq 0$

$y \geq 0 \qquad\qquad 2y \geq 0$

$x + 3y \geq 2$

lower bound $B = 2$

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min $\quad px + qy$

s.t. $\quad x + y \geq 2 \qquad ax + ay \geq 2a$

$x \geq 0 \qquad bx \geq 0$

$y \geq 0 \qquad cy \geq 0$

$(a+b)x + (a+c)y \geq 2a$

$px + qy \geq 2a$

as long as $a, b, c \geq 0$

$a + b = p$

$a + c = q$

lower bound $B = 2a$ provided

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min $\quad px + qy$

s.t. $\quad x \geq 0 \qquad ax \geq 0 \qquad a \geq 0$

$y \leq 1, \; -y \geq -1 \qquad -by \geq -b \qquad b \geq 0$

$3x + y = 2, \qquad 3cx + cy = 2c$

$(a+3c)x + (-b+c)y \geq -b + 2c \qquad a + 3c = p$

$\underset{p}{\underset{\shortparallel}{}}x \qquad \underset{q}{\underset{\shortparallel}{}}y \geq -b + 2c \qquad -b + c = q$

②

$$\begin{array}{lll|l} \min & c^T x & & \\ \text{st.} & Ax = b & u & \\ & Gx \le h & v & v \geq 0 \end{array}$$

x feasible $\Longrightarrow$

$$u^T(Ax-b) + v^T(Gx-h) \le 0$$

$$\sum v_i \cdot (Gx-h)_i$$

$$(A^T u + G^T v)^T x - b^T u - h^T v \le 0$$

$$(\underbrace{-A^T u - G^T v}_{\overset{\shortparallel}{c}})^T x \geq -b^T u - h^T v$$

$$c^T x \geq \underline{\underline{-b^T u - h^T v}}$$

$$\begin{array}{lll|l} \min & c^T x & & \\ \text{st} & Ax = b & u & \\ & Gx \le h & v & v \geq 0 \end{array}$$

x feasible $\Longrightarrow$

$$c^T x \geq c^T x + \underbrace{\underbrace{u^T(Ax-b)}_{=0} + \underbrace{v^T(Gx-h)}_{\le 0}}_{\le 0}$$

denote $C = \{x : Ax = b,\ Gx \le h\}$ feasible set

$$f^* = \min_{x \in C} c^T x \geq \min_{x \in C} c^T x + u^T(Ax-b) + v^T(Gx-h)$$

$$\geq \underbrace{\min_{x} c^T x + u^T(Ax-b) + v^T(Gx-h)}_{:= g(u,v)}$$

③

shown $f^\star \geq g(u,v)$ provided $v \geq 0$

Dual problem: $\max_{u,v} \; g(u,v)$

st. $v \geq 0$

Have by construction: $g^\star \leq f^\star$.

What is g for our LP? :

$$g(u,v) = \min_x \; c^T x + u^T(Ax-b) + v^T(Gx-h)$$

$$= \min_x \; (c + A^T u + G^T v)^T x - b^T u - h^T v$$

$$= \begin{cases} -\infty & \text{if } c + A^T u + G^T v \neq 0 \\ -b^T u - h^T v & \text{if } c + A^T u + G^T v = 0 \end{cases}$$

Therefore

$$\begin{array}{ll} \max_{u,v} & g(u,v) \\ \text{st.} & v \geq 0 \end{array} \iff \underbrace{\begin{array}{ll} \max_{u,v} & -b^T u - h^T v \\ \text{st.} & c = -A^T u - G^T v \\ & v \geq 0 \end{array}}_{\text{dual LP.}}$$

Foreshadowing:

$$\begin{array}{lll|l} \min & f(x) & & \\ \text{s.t.} & Ax = b & u & \\ & Gx \le h & v & v \ge 0 \end{array}$$

$$u^T(Ax-b) + v^T(Gx-h) \le 0$$

Strategy #1: let's get this looking like

$$f(x) + \text{something} \le 0$$

not going to work unless $f(x)$ is linear

Strategy #2: works for any $f(x)$!

$$x^T P y = (P^T x)^T y$$
$$= \Sigma (P^T x)_i \, y_i$$

$$= (P^T x)_1 y_1 + (P^T x)_2 y_2 + \ldots +$$

if $(P^T x)_1$ is largest, take $y_1 = 1$.

$$x^T P y = \Sigma \, x_j (Py)_j$$

$$\min \{ x^T P y : x \ge 0, \ 1^T x = 1 \} = \min_j (Py)_j$$

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$$\min_{x} \max_{i=1..m} (P^T x)_i$$

s.t. $x \geq 0$, $1^T x = 1$

$\Longleftrightarrow$

$$\min_{x,t} \quad t$$

s.t. $x \geq 0$, $1^T x = 1$

$P^T x \leq t.$

$$t - u^T x + v(1 - 1^T x) + y^T(P^T x - t1)$$

$$= (-u - v1 + Py)^T x + (1 - y^T 1)\cdot t + v$$