①

Assume $y \in \partial f(x)$

$$\min_z \; f(z) - z^T y$$

optimal z : $0 \in \partial f(z) - y$

$\iff$ $y \in \partial f(z)$

Thus x must minimize $f(z) - z^T y$ over z

$\iff$ maximize $z^T y - f(z)$ over z

Recall: $f^*(y) = \max_z \; z^T y - f(z) := \max_z \; h_z(y)$

$$\partial f^*(y) = \mathrm{cl}\Big(\mathrm{conv}\Big(\bigcup_{\substack{z:\\ z \text{ achieves} \\ \max}} \{z\}\Big)\Big)$$

Hence $x \in \partial f^*(y)$

Assume $x \in \partial f^*(y)$

From above, $y \in \partial f^{**}(x)$

Hence $y \in \partial f(x)$ ✓

$x \in \partial f^*(-A^T u)$

$\iff x \in \operatorname{argmin}_z \; f(z) - (-A^T u)^T z$

$\iff x \in \operatorname{argmin} \; f(z) + u^T A z$

strong convexity:

$$g(y) \geq g(x) + \nabla g(x)^T (y-x) + \frac{d}{2} \|y-x\|_2^2 \quad \text{all } x, y$$

at minimizer x of g.

$$g(y) \geq g(x) + \frac{d}{2} \|y-x\|_2^2 \quad \text{all } y$$

let's apply this to

$$g(x) = f(x) - u^T x \quad \text{minimized at } x_u = \nabla f^*(u)$$

$$g(x) = f(x) - v^T x \quad \text{minimized at } x_v = \nabla f^*(v)$$

adding ineq's together:

$$\cancel{f(x_v)} - u^T x_v + \cancel{f(x_u)} - v^T x_u$$

$$\geq \cancel{f(x_u)} - u^T x_u + \cancel{f(x_v)} - v^T x_v + d \|x_u - x_v\|_2^2$$

$$\begin{aligned} d \|x_u - x_v\|_2^2 &\leq u^T x_v + u^T x_u + v^T x_v + v^T x_u \\ &= (u-v)^T (x_u - x_v) \\ &\leq \|u-v\|_2 \|\cancel{x_u - x_v}\|_2 \end{aligned}$$

$$\|x_u - x_v\|_2 \leq \frac{1}{d} \|u-v\|_2$$

$$Ax = b \iff \sum_{i=1}^{B} A_i x_i = b$$

$$A = \left[A_1 \,|\, A_2 \,|\, \cdots \,|\, A_B \right] \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_B \end{bmatrix}$$

③

$$\min_x \; f(x) + u^T A x$$

$$\Longleftrightarrow \min_x \; \sum f_i(x_i) + u^T \left(\sum A_i x_i \right)$$

$$\Longleftrightarrow \min_x \; \sum \left(f_i(x_i) + u^T A_i x_i \right)$$

$a_j^T x \le b_j$ constraint on resource j

$u_j \ge 0$ price of resource j

$$\min_x \; f(x) + \frac{\rho}{2} \| Ax + Bz^{(k-1)} - c + w^{(k-1)} \|_2^2$$

$$\Longleftrightarrow \min_x \; f(x) + \frac{\rho}{2} \| Ax + Bz^{(k-1)} - c \|_2^2 + \cancel{\| w^{(k-1)} \|_2^2}$$

$$+ \underbrace{2 \frac{\rho}{2} w^{(k-1)}}_{\substack{= \rho w^{(k-1)} \\ = u^{(k-1)}}}{}^T (Ax + Bz^{(k-1)} - c)$$

$$x: \quad \arg\min_x \; I_C(x) + \frac{\rho}{2} \| x - z^{(k-1)} + w^{(k-1)} \|_2^2$$

$$= P_C(z^{(k-1)} - w^{(k-1)})$$

$$z: \quad P_D(x^{(k)} + w^{(k-1)})$$

$$w: \quad w^{(k)} = w^{(k-1)} + x^{(k)} - z^{(k)}$$