Special case when strong duality holds

Slater condn: $\exists\ \hat{x} \in \text{int}(D)$

$$h(\hat{x}) < 0 \quad \& \quad l(\hat{x}) = 0$$

then strong duality holds $\underline{\underline{p = d}}$

(Primal) barrier problem

$$\min_x \ c^T x - \tau \sum_{i=1}^{n} \log(x_i)$$

$$Ax = b$$

$$L(x, y) = c^T x - \tau \sum \log(x_i) + y^T (b - Ax)$$

Dual problem

$$\max_y \ \min_x \ (c - A^T y)^T x - \tau \sum \log(x_i) + \underset{\sim}{b^T y}$$

$$\min_x \ \underbrace{(c - A^T y)}_{s := c - A^T y}{}^T x - \tau \sum \log(x_i)$$

decouples into

$$\min_x \sum \big(\underbrace{s_i x_i - \tau \log(x_i)}_{\substack{\text{minimized} \\ \text{at } x_i = \frac{\tau}{s_i}}} \big)$$

Dual Problem:

$$\max_{y,s} \; b^T y + n\tau - \tau \sum \log\left(\frac{\tau}{s_i}\right)$$
$$A^T y + s = c$$

$$\Longleftrightarrow \quad \max_{y,s} \; b^T y + \tau \sum \log s_i + n\tau - n\tau \log \tau$$

Observe: If $(x^*(\tau), u^*(\tau), v^*(\tau))$ solve KKT condns for barrier problem (slide 12)

Duality gap $f(x^*(\tau)) - \underbrace{\min_x L(x, u^*(\tau), v^*(\tau))}_{\text{dual function eval at } (u^*(\tau), v^*(\tau))}$

$$= f(x^*(\tau)) - L(x^*(\tau), u^*(\tau), v^*(\tau))$$
$$= f(x^*(\tau)) - \left[f(x^*(\tau)) + u^*(\tau)^T h(x^*(\tau)) + v^*(\tau)^T \underbrace{(b - Ax^*(\tau))}_{0} \right]$$

$$= m\tau$$

Fenchel duality

Primal: $\; f(x) + g(z)$

$$Ax = z$$

$$L(x,z,v) = f(x) + g(z) + v^T(z - Ax)$$

Dual

$$\max_{v} \; \min_{x,z} \left[v^T z + g(z) - (A^T v)^T x + f(x) \right]$$

Recall $\quad f^*(s) := \max_x \left(s^T x - f(x) \right)$

Dual $\quad \max_v \; -f^*(A^T v) - g^*(-v)$

Conic programming

Primal $\quad \min_x \; c^T x \quad \Longleftrightarrow \quad \min_x f(x) + g(Ax)$

$$Ax = b$$

$$x \in K$$

for $\; f(x) = c^T x + I_K(x)$

$\qquad g(z) = I_{\{b\}}(z)$

Observe: $\quad f(x) + g(Ax) = \begin{cases} c^T x & \text{if } \begin{matrix} Ax = b \\ x \in K \end{matrix} \\ +\infty & \text{o.w.} \end{cases}$

Recall: $I_K^*(s) = \max_x s^T x - I_K(x)$

If K closed convex cone $= \max_{x \in K} s^T x$

$$= \begin{cases} 0 & s \in -K^* \\ +\infty & s \notin -K^* \end{cases}$$

$$= I_{-K^*}(s)$$

Strong duality may fail in SDP

$$\min \ 2X_{12} \qquad\qquad \min \ C \cdot X$$

$$\begin{bmatrix} 0 & X_{12} \\ X_{12} & X_{22} \end{bmatrix} \succeq_C 0 \iff \begin{array}{l} A \cdot X = b \\ X \succeq_C 0 \end{array}$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ b = 0$$

dual $\max \ 0 \cdot y$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} y + S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S \succeq_C 0$$

Constraint $\iff \begin{bmatrix} -y & 1 \\ 1 & 0 \end{bmatrix} \succeq_C 0$