$$f(x) = \frac{1}{2} x^T Q x \quad , \quad Q \succ 0$$

$$\Rightarrow \quad f^*(y) = \frac{1}{2} y^T Q^{-1} y$$

$$f(x) = \frac{1}{2} x^T Q x + b^T x \quad , \quad Q \succ 0$$

$$\Rightarrow \quad f^*(y) = \frac{1}{2} (y-b)^T Q^{-1} (y-b)$$

(Pure)

Newton's method for $\min_x f(x)$

$$x^+ = x - \nabla^2 f(x)^{-1} \nabla f(x)$$

$$\min_y \left[f(x) + \nabla f(x)^T (y-x) + \frac{1}{2} (y-x)^T \nabla^2 f(x) (y-x) \right]$$

$$\nabla f(x) + \nabla^2 f(x) (y-x) = 0$$

$$\Rightarrow \quad y = x - \nabla^2 f(x)^{-1} \nabla f(x)$$

If $\quad F(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$

$$F'(x) = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_n(x)^T \end{bmatrix}$$

$$F(x) = x^2 - 2$$

Newton's method: $x^+ = x - (2x)^{-1}(x^2 - 2)$

$$= \frac{x}{2} + \frac{1}{x}$$

Recall: NM for root-finding $F(x) = 0$

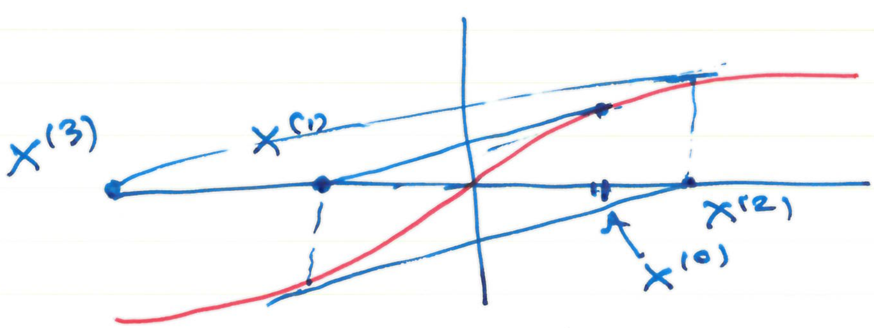
$$x^+ = x - F'(x)^{-1} F(x)$$

Linear convergence: $x^{(k)} \to x^*$ linearly

$$\iff \|x^{(k+1)} - x^*\| \leq c\,\|x^{(k)} - x^*\|$$

for some $c \in (0,1)$

Newton's method may not converge

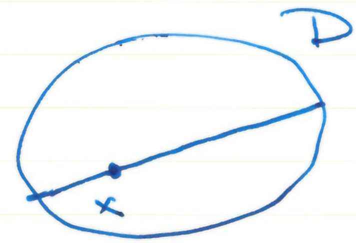


$f : D \to \mathbb{R}$ $\qquad D \subseteq \mathbb{R}^n$ open & convex

f is self-concordant if $\forall\, x \in D,\ d \in \mathbb{R}^n$

$$t \longmapsto f(x+td)$$

is self-concordant



Example of sc function $\quad f : \mathbb{R}_+ \to \mathbb{R}$

$$f(t) = -\log(t)$$

$$f'(t) = -\frac{1}{t}, \quad f''(t) = \frac{1}{t^2}, \quad f'''(t) = -\frac{2}{t^3}$$

$$|f'''(t)| = \frac{2}{t^3} = 2\, f''(t)^{3/2}$$

Another example

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$f(x) = -\sum_{j=1}^{m} \log(a_j^T x - b_j) \quad \text{is sc}$$

$$f : \{x : Ax > b\} \to \mathbb{R}$$

level curves of f