Dual Methods and ADMM

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Last time: case study of generalized lasso

We studied generalized lasso problems:

$$\min_{\beta} \; f(\beta) + \lambda \|D\beta\|_1$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth, convex function and $D \in \mathbb{R}^{m \times n}$ is a penalty matrix

- We derived its dual problem, and considered applying all of the algorithms we've learned so far to both its primal and its dual
- We saw that different algorithms had different strengths, and were suitable for different situations

For the remainder of the course, we will study advanced methods, which go beyond the first- and second-order paradigms

Conjugate functions

Reminder: given $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$f^*(y) = \max_x \; y^T x - f(x)$$

is called its conjugate

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \; f(x) - y^T x$$

- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \iff y \in \partial f(x) \iff x \in \operatorname*{argmin}_z \; f(z) - y^T z$$

- If f is strictly convex f, then $\nabla f^*(y) = \operatorname*{argmin}_z \; f(z) - y^T z$

Outline

Today:

- Dual (sub)gradient methods
- Dual decomposition
- Augmented Lagrangians
- ADMM

Dual (sub)gradient methods

What if we can't derive dual (conjugate) in closed form, but want to utilize dual relationship? Turns out we can still use dual-based subradient or gradient methods

Example: consider the problem

$$\min_x \ f(x) \ \text{ subject to } \ Ax = b$$

Its dual problem is

$$\max_u \ -f^*(-A^T u) - b^T u$$

where f^* is conjugate of f. Defining $g(u) = f^*(-A^T u)$, note that $\partial g(u) = -A\partial f^*(-A^T u)$, and recall

$$x \in \partial f^*(-A^T u) \iff x \in \operatorname*{argmin}_z \ f(z) + u^T A z$$

Therefore the dual subgradient method (for maximizing the dual objective) starts with an initial dual guess $u^{(0)}$, and repeats for $k = 1, 2, 3, \ldots$

$$x^{(k)} \in \operatorname*{argmin}_x \; f(x) + (u^{(k-1)})^T Ax$$
$$u^{(k)} = u^{(k-1)} + t_k(Ax^{(k-1)} - b)$$

where t_k are step sizes, chosen in standard ways

Recall that if f is strictly convex, then f^* is differentiable, and so we get dual gradient ascent, which repeats for $k = 1, 2, 3, \ldots$

$$x^{(k)} = \operatorname*{argmin}_x \; f(x) + (u^{(k-1)})^T Ax$$
$$u^{(k)} = u^{(k-1)} + t_k(Ax^{(k-1)} - b)$$

(difference is that each $x^{(k)}$ is unique, here). Proximal gradients and acceleration carry through in similar manner

Covergence analysis

First recall that if f strongly convex with parameter d, then ∇f^* Lipschitz with parameter $1/d$

Proof: if f strongly convex and x is its minimizer, then

$$f(y) \geq f(x) + \frac{d}{2}\|y - x\|_2, \quad \text{for all } y$$

Hence defining $x_u = \nabla f^*(u)$, $x_v = \nabla f^*(v)$,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{d}{2}\|x_u - x_v\|_2^2$$
$$f(x_u) - v^T x_u \geq f(x_v) - v^T x_v + \frac{d}{2}\|x_u - x_v\|_2^2$$

Adding these together, using Cauchy-Schwartz, and rearranging shows that

$$\|x_u - x_v\|_2 \leq \frac{1}{d} \cdot \|u - v\|_2$$

Applying what we know about gradient descent: if f is strongly convex with parameter d, then dual gradient ascent with constant step size $t_k \leq d$ converges at rate $O(1/\epsilon)$

Is this a slow or fast rate, compared to what we would get out of primal gradient descent? It's actually essentially the same

- When f is strongly convex, primal gradient descent converges at rate $O(1/\epsilon)$. But if we further assume that ∇f is Lipschitz, then we get the linear rate $O(\log(1/\epsilon))$
- Note: the converse of the statement on the last slide is also true: ∇f^* being Lipschitz with parameter $1/d$ implies that f is strongly convex with parameter d
- Hence assume $f^{**} = f$. When f has Lipschitz gradient and is strongly convex, the same is true about f^*, and dual gradient ascent also converges at the linear rate $O(\log(1/\epsilon))$

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly

$$A = [A_1, \ldots A_B], \quad \text{where } A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of (sub)gradient:

$$x^+ \in \operatorname*{argmin}_x \sum_{i=1}^B f_i(x_i) + u^T Ax$$

$$\iff x_i^+ \in \operatorname*{argmin}_{x_i} f_i(x_i) + u^T A_i x_i, \quad i = 1, \ldots B$$

i.e., minimization decomposes into B separate problems

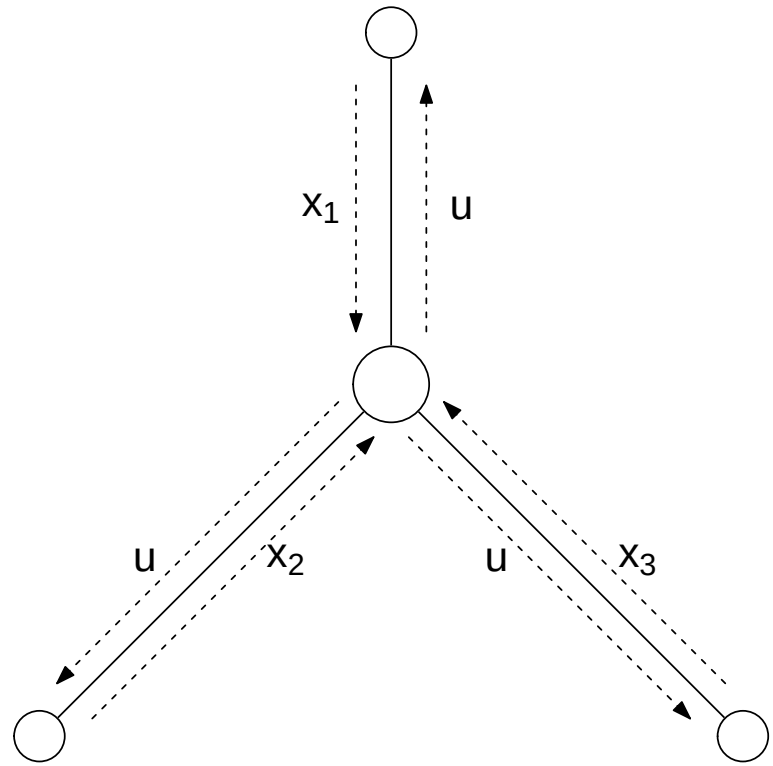
Dual decomposition algorithm: repeat for $k = 1, 2, 3, \ldots$

$$x_i^{(k)} \in \operatorname*{argmin}_{x_i} \; f_i(x_i) + (u^{(k-1)})^T A_i x_i, \quad i = 1, \ldots B$$

$$u^{(k)} = u^{(k-1)} + t_k \Big(\sum_{i=1}^{B} A_i x_i^{(k-1)} - b \Big)$$

Can think of these steps as:

- Broadcast: send u to each of the B processors, each optimizes in parallel to find x_i
- Gather: collect $A_i x_i$ from each processor, update the global dual variable u



Example with inequality constraints:

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \leq b$$

Dual decomposition (projected subgradient method) repeats for $k = 1, 2, 3, \ldots$

$$x_i^{(k)} \in \operatorname*{argmin}_{x_i} \; f_i(x_i) + (u^{(k-1)})^T A_i x_i, \quad i = 1, \ldots B$$

$$v^{(k)} = u^{(k-1)} + t_k \Big(\sum_{i=1}^B A_i x_i^{(k-1)} - b \Big)$$

$$u^{(k)} = (v^{(k)})_+$$

where $(\cdot)_+$ is componentwise thresholding, $(u_+)_i = \max\{0, u_i\}$

Price coordination interpretation (from Vandenberghe's lecture notes):

- Have B units in a system, each unit chooses its own decision variable x_i (how to allocate its goods)
- Constraints are limits on shared resources (rows of A), each component of dual variable u_j is price of resource j
- Dual update:

$$u_j^+ = (u_j - ts_j)_+, \quad j = 1, \ldots m$$

  where $s = b - \sum_{i=1}^{B} A_i x_i$ are slacks
  - Increase price u_j if resource j is over-utilized, $s_j < 0$
  - Decrease price u_j if resource j is under-utilized, $s_j > 0$
  - Never let prices get negative

Augmented Lagrangian

Disadvantage of dual methods: require strong conditions to ensure primal iterates converge to solutions. Convergence properties can be improved by utilizing augmented Lagrangian. Transform primal:

$$\begin{array}{ll} \min\limits_{x} & f(x) + \dfrac{\rho}{2}\|Ax - b\|_2^2 \\ \text{subject to} & Ax = b \end{array}$$

Clearly extra term $(\rho/2) \cdot \|Ax - b\|_2^2$ does not change problem. Use dual gradient ascent: repeat for $k = 1, 2, 3, \ldots$

$$\begin{aligned} x^{(k)} &= \operatorname*{argmin}_x \; f(x) + (u^{(k-1)})^T Ax + \frac{\rho}{2}\|Ax - b\|_2^2 \\ u^{(k)} &= u^{(k-1)} + \rho(Ax^{(k-1)} - b) \end{aligned}$$

(When, e.g., A has full column rank, primal is guaranteed strongly convex)

Notice step size choice $t_k = \rho$, for all k, in dual gradient ascent. Why? Since $x^{(k)}$ minimizes $f(x) + (u^{(k-1)})^T Ax + \frac{\rho}{2}\|Ax - b\|_2^2$ over x, we have

$$\begin{aligned} 0 &\in \partial f(x^{(k)}) + A^T\Big(u^{(k-1)} + \rho(Ax^{(k)} - b)\Big) \\ &= \partial f(x^{(k)}) + A^T u^{(k)} \end{aligned}$$

This is the stationarity condition for the original primal problem; can show under mild conditions that $Ax^{(k)} - b$ approaches zero (i.e., primal iterates approach feasibility), hence in the limit KKT conditions are satisfied and $x^{(k)}, u^{(k)}$ approach optimality

Advantage: much better convergence properties. Disadvantage: lose decomposability! (Separability is compromised by augmented Lagrangian ...)

Alternating direction method of multipliers

Alternating direction method of multipliers or ADMM: the best of both worlds!

I.e., good convergence properties of augmented Lagrangians, along with decomposability

Consider minimization problem

$$\min_x \ f_1(x_1) + f_2(x_2) \ \text{ subject to } \ A_1x_1 + A_2x_2 = b$$

As before, we augment the objective

$$\begin{aligned} \min_x \quad & f_1(x_1) + f_2(x_2) + \frac{\rho}{2}\|A_1x_1 + A_2x_2 - b\|_2^2 \\ \text{subject to} \quad & A_1x_1 + A_2x_2 = b \end{aligned}$$

Write the augmented Lagrangian as

$$L_\rho(x_1, x_2, u) = f_1(x_1) + f_2(x_2) + u^T(A_1x_1 + A_2x_2 - b) + \frac{\rho}{2}\|A_1x_1 + A_2x_2 - b\|_2^2$$

Now ADMM repeats the steps, for $k = 1, 2, 3, \ldots$

$$\begin{aligned}
x_1^{(k)} &= \operatorname*{argmin}_{x_1} \; L_\rho(x_1, x_2^{(k-1)}, u^{(k-1)}) \\
x_2^{(k)} &= \operatorname*{argmin}_{x_2} \; L_\rho(x_1^{(k)}, x_2, u^{(k-1)}) \\
u^{(k)} &= u^{(k-1)} + \rho(A_1x_1^{(k)} + A_2x_2^{(k)} - b)
\end{aligned}$$

Note that the usual method of multipliers would have replaced the first two steps by

$$(x_1^{(k)}, x_2^{(k)}) = \operatorname*{argmin}_{x_1, x_2} \; L_\rho(x_1, x_2, u^{(k-1)})$$

Convergence guarantees

Under modest assumptions on f_1, f_2 (these do not require A_1, A_2 to be full rank), the ADMM iterates satisfy, for any $\rho > 0$:

- Residual convergence: $r^{(k)} = A_1 x_1^{(k)} - A_2 x_2^{(k)} - b \to 0$ as $k \to \infty$, i.e., primal iterates approach feasibility
- Objective convergence: $f_1(x_1^{(k)}) + f_2(x_2^{(k)}) \to f^\star$, where $f^\star$ is the optimal primal criterion value
- Dual convergence: $u^{(k)} \to u^\star$, where $u^\star$ is a dual solution

For details, see Boyd et al. (2010). Note that we do not generically get primal convergence, but this can be shown under more assumptions

Convergence rate: not known in general, but known in few special cases. Rough consensus seems to be that it behave like first-order methods

Scaled form

It is often easier to express the ADMM algorithm in a scaled form, where we replace the dual variable u by a scaled variable $w = u/\rho$. In this parametrization, the ADMM steps are

$$
\begin{aligned}
x_1^{(k)} &= \underset{x_1}{\operatorname{argmin}} \; f_1(x_1) + \frac{\rho}{2}\|A_1 x_1 + A_2 x_2^{(k-1)} - b + w^{(k-1)}\|_2^2 \\
x_2^{(k)} &= \underset{x_2}{\operatorname{argmin}} \; f_2(x_2) + \frac{\rho}{2}\|A_1 x_1^{(k)} + A_2 x_2 - b + w^{(k-1)}\|_2^2 \\
w^{(k)} &= w^{(k-1)} + A_1 x_1^{(k)} + A_2 x_2^{(k)} - b
\end{aligned}
$$

Note that here the kth iterate $w^{(k)}$ is just given by a running sum of residuals:

$$
w^{(k)} = w^{(0)} + \sum_{i=1}^{k} \left(A_1 x_1^{(i)} + A_2 x_2^{(i)} - b\right)
$$

Practicalities and tricks

Practical experience shows that ADMM usually obtains a relatively accurate solution in a handful of iterations, but requires a very large number of iterations for a highly accurate solution. This is more evidence that it behaves like a first-order method

Choice of ρ can greatly influence practical convergence of ADMM:

- ρ too large $\rightarrow$ not enough emphasis on minimizing $f_1 + f_2$
- ρ too small $\rightarrow$ not enough emphasis on feasibility

Boyd et al. (2010) give a strategy for varying ρ that can be useful in practice (but does not have convergence guarantees)

Like deriving duals, transforming a problem into that ADMM can handle often requires a bit of trickery (and different forms can lead to different algorithms)

Example: alternating projections

Consider finding a point in intersection of convex sets $C, D \subseteq \mathbb{R}^n$, i.e., solving

$$\min_x \ 1_C(x) + 1_D(x)$$

To get this into ADMM form, we express it as

$$\min_{x,z} \ 1_C(x) + 1_D(z) \ \text{ subject to } \ x - z = 0$$

Each ADMM cycle involves two projections:

$$\begin{aligned} x^{(k)} &= \operatorname*{argmin}_x \ P_C\big(z^{(k-1)} - w^{(k-1)}\big) \\ z^{(k)} &= \operatorname*{argmin}_z \ P_D\big(x^{(k)} + w^{(k-1)}\big) \\ w^{(k)} &= w^{(k-1)} + x^{(k)} - z^{(k)} \end{aligned}$$

This is like the classical alternating projections method, but now with a dual variable w. It is much more efficient

Example: generalized lasso regression

Given the usual $y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times p}$, and an additional $D \in \mathbb{R}^{m \times p}$, the generalized lasso problem solves

$$\min_{\beta \in \mathbb{R}^p} \ \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \|D\beta\|_1$$

This computationally harder than the lasso problem (with $D = I$); recall our study on algorithms for this problem. We can rewrite as

$$\min_{\beta \in \mathbb{R}^p,\, \alpha \in \mathbb{R}^m} \ \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \|\alpha\|_1 \ \text{ subject to } \ D\beta - \alpha = 0$$

and ADMM gives us a simple algorithm for the generalized lasso:

$$\begin{aligned}
\beta^{(k)} &= (X^T X + \rho D^T D)^+ \big(X^T y + \rho D^T (\alpha^{(k-1)} - w^{(k-1)})\big) \\
\alpha^{(k)} &= S_{\lambda/\rho}(D\beta^{(k)} + w^{(k-1)}) \\
w^{(k)} &= w^{(k-1)} + D\beta^{(k)} - \alpha^{(k)}
\end{aligned}$$

Example: sum-of-norms regularization

Now consider

$$\min_{\beta \in \mathbb{R}^p} \ \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \sum_{g=1}^{G} \|\beta_{I_g}\|_2$$

where each $\beta_{I_g} \in \mathbb{R}^{|I_g|}$ is a sub-block of the full coefficient vector β. Called a group lasso problem, or a sum-of-norms regularization problem when we generalize the ℓ_2 norm above. Rewrite as

$$\min_{\beta \in \mathbb{R}^p,\, \alpha \in \mathbb{R}^p} \ \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \sum_{g=1}^{G} \|\alpha_{I_g}\|_2 \ \text{ subject to } \ \beta - \alpha = 0$$

andf ADMM updates become:

$$\begin{aligned}
\beta^{(k)} &= (X^T X + \rho I)^{-1}\big(X^T y + \rho(\alpha^{(k-1)} - w^{(k-1)})\big) \\
\alpha_{I_g}^{(k)} &= R_{\lambda/\rho}\big(\beta_{I_g}^{(k)} + w_{I_g}^{(k-1)}\big), \quad g = 1, \ldots G \\
w^{(k)} &= w^{(k-1)} + \beta^{(k)} - \alpha^{(k)}
\end{aligned}$$

Notes:

- The matrix $X^T X + \rho I$ is always invertible, regardless of X
- If we take its factorization (say QR), in $O(p^3)$ flops, then each subsequent solve takes $O(p^2)$ flops
- The shrinkage operator R_t is defined as

$$R_t(x) = \left(1 - \frac{t}{\|x\|_2}\right)_+ x$$

- Similar steps can be performed for a sum-of-norms problem, as long as can solve for the prox operator of the individual norms
- An ADMM algorithm can also be developed for the case of overlapping groups (which is otherwise quite a hard problem to optimize!). See Boyd et al. (2010)

References

- S. Boyd and N. Parikh and E. Chu and B. Peleato and J. Eckstein (2010), "Distributed optimization and statistical learning via the alternating direction method of multipliers"
- L. Vandenberghe, Lecture Notes for EE 236C, UCLA, Spring 2011-2012