$$\begin{aligned} \min\ & x + 3y \\ \text{s.t. } & x+y \geq 2 \\ & x \geq 0 \\ & y \geq 0 \quad 2y \geq 0 \end{aligned} \Bigg\}\ x + 3y \geq 2$$

bound is $B = 2$

$$A = \begin{aligned} \min\ & px + qy \\ \text{s.t. } & x+y \geq 2 \quad \cdot\ a \geq 0 \\ & x \geq 0 \quad \cdot\ b \geq 0 \\ & y \geq 0 \quad \cdot\ c \geq 0 \end{aligned} \Bigg\}\ \begin{aligned} & ax + ay \geq 2a \\ & bx \geq 0 \\ & cy \geq 0. \end{aligned}$$

$$\underbrace{(a+b)}_{p} x + \underbrace{(a+c)}_{q} y \geq 2a$$

$$B = \begin{aligned} \max\ & 2a \\ \text{such that } & a+b = p \\ & a+c = q \\ & a,b,c \geq 0. \end{aligned}$$

by construction we have $A \geq B$

largest or tightest lower bound

$$\begin{aligned} \min\ & px + qy \\ \text{s.t. } & x \geq 0 \quad a \geq 0 \\ & -y \geq -1 \quad b \geq 0 \\ & 3x + y = 2 \quad c \end{aligned} \Bigg\}\ \begin{aligned} & ax \geq 0 \\ & -by \geq -b \\ & 3cx + cy = 2c \end{aligned}$$

$$\begin{aligned} \max\ & -b + 2c \\ \text{s.t. } & a + 3c = p \\ & -b + c = q \\ & a, b \geq 0 \end{aligned}$$

$$\underbrace{(a+3c)}_{p} x + \underbrace{(-b+c)}_{q} y \geq -b + 2c$$

$x \in \mathbb{R}^n$, A $m \times n$ G $r \times n$

primal LP
$$\begin{cases} \min_x & c^T x \\ \text{s.t.} & Ax = b \quad u \in \mathbb{R}^m \\ & Gx \le h \quad v \in \mathbb{R}^r,\ v \geq 0 \end{cases}$$

$$u^T A x = u^T b$$
$$v^T G x \le v^T h$$
$$u^T (Ax - b) + v^T (Gx - h) \le 0$$
$$(A^T u + G^T v)^T x - b^T u - h^T v \le 0$$
$$\underbrace{(-A^T u - G^T v)^T}_{c} x \geq -b^T u - h^T z$$

$$\left.\begin{array}{ll} \max_{u,v} & -b^T u - h^T z \\ \text{s.t.} & -A^T u - G^T v = c \\ & v \geq 0 \end{array}\right\} \text{dual LP}$$

$$\max_{f \in \mathbb{R}^{|E|}} \sum_{(s,j) \in E} f_{sj}$$

$$\begin{array}{llll} \text{s.t.} & -f_{ij} \le 0 & \text{all } ij & a_{ij} \geq 0 \\ & -f_{ij} - c_{ij} \le 0 & & b_{ij} \geq 0 \\ & \sum_{(i,k) \in E} f_{ik} = \sum_{(k,j) \in E} kj & \text{all } k \neq s,t & x_k \end{array}$$

$$\sum_{(i,j) \in E} \left(-a_{ij} f_{ij} + b_{ij} (f_{ij} - c_{ij})\right) + \sum_{k \neq s,t} x_k \left(\sum f_{ik} - \sum f_{kj} \right)$$
$$\le 0.$$

$$\sum_{i,j} \underline{M_{ij} f_{ij}} \le \boxed{\sum_{i,j} b_{ij} c_{ij}}$$

$$M_{sj} = -a_{sj} + b_{sj} + x_j \qquad \text{want} = 1$$
$$M_{it} = -a_{it} + b_{it} - x_i \qquad \text{want} = 0$$
$$M_{ij} = -a_{ij} + b_{ij} - x_i + x_j \qquad \text{want} = 0$$
$i \neq s,\ j \neq t$

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dual LP

$$\min_{a,b,x} \quad \sum b_{ij} c_{ij}$$

s.t.

$$b_{sj} - a_{sj} + x_j = 1 \qquad \text{all } j$$
$$b_{it} - a_{it} - x_i = 0 \qquad \text{all } i$$
$$b_{ij} - a_{ij} - x_i + x_j = 0 \qquad \text{all } i,j$$
$$a, b \geq 0$$

$a^T x = b + y,\ y \geq 0 \iff a^T x \geq b$

$\Updownarrow$

$$b_{sj} + x_j = 1 + a_{sj} \geq 1$$
$$b_{it} - x_i = a_{it} \geq 0$$
$$b_{ij} - x_i + x_j = a_{ij} \geq 0 \qquad \text{all } i,j \quad i \neq s,\ j \neq t$$

dual LP

$$\min_{b,x} \quad \sum b_{ij} c_{ij}$$

s.t.

$$b_{ij} + x_j - x_i \geq 0 \qquad \text{all } i,j$$
$$b \geq 0,\ x_s = 1,\ x_t = 0.$$

suppose that at dual solution, we had

$x_i \in \{0, 1\}$ all i

$A = \{i : x_i = 1\}$
$B = \{j : x_j = 0\}$

then

$b_{ij} \geq x_i - x_j \implies b_{ij} = 1$ if $i \in A, j \in B$
$\qquad\qquad\qquad\qquad = 0$ otherwise

criterion $\sum b_{ij} c_{ij}$
equals value of the cut (A, B)

dual is LP relaxation of min cut problem

$$f^* = \min_{x \in C} c^T x \geq \min_{x \in C} L(x, u, v) \leftarrow \text{Lagrangian}$$

$$\geq \min_{x} L(x, u, v)$$

$$:= g(u, v) \quad \text{Lagrange dual function}$$

$$L(x, u, v) = (A^T u + c + G^T v)^T x - b^T u - h^T v$$

$$g(u, v) = \min_x L(x, u, v)$$

$$= \begin{cases} -\infty & \text{if } A^T u + G^T v + c \neq 0 \\ -b^T u - h^T v & \text{else} \end{cases}$$