for any dual $u, v$ (i.e. $u \geq 0$)

$$f^* = \min_{x \in C} f(x) \geq \min_{x \in C} L(x,u,v)$$
$$\geq \min_{x} L(x,u,v)$$
$$:= g(u,v) \quad \text{Lagrange dual}$$

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QP: $\min_{x} \quad \frac{1}{2} x^T Q x + c^T x$ where $Q \succ 0$.

st. $\underset{v}{Ax = b}, \ \underset{u}{-x \leq 0}$

Lagrangian:

$$L(x,u,v) = \frac{1}{2} x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

suppose $u \geq 0$, $v$.

Lagrange dual function:

$$g(u,v) = \min_{x} \frac{1}{2} x^T Q x + (c - u + A^T v)^T x - b^T v$$

minimized at $x = -Q^{-1}(c - u + A^T v)$

then

$$g(u,v) = -\frac{1}{2}(c - u + A^T v)^T Q^{-1} (c - u + A^T v) - b^T v$$

$ax^2 + bx + c$
minimized at $x = \frac{-b}{2a}$

$$\max \quad \underbrace{-f(x) - \sum u_i h_i(x) - \sum v_j \ell_j(x)}_{l_x(u,v)}$$

affine for each $x$

convex for each $x$

$$\underbrace{\max_{x} \; l_x(u,v)}_{\text{convex}} = -g(u,v)$$

hence $g(u,v)$ is concave

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$$\min \quad \frac{1}{2}\|\beta\|_2^2 + C\sum \xi_i$$

$$\text{st} \quad -\xi \le 0 \qquad v \ge 0$$

$$1 - \xi_i - y_i(x_i^T\beta + \beta_0) \le 0 \qquad w_i \ge 0$$

$$L(\beta, \beta_0, \xi, v, w) = \frac{1}{2}\|\beta\|_2^2 + C\sum \xi_i - \sum v_i \xi_i + \sum w_i\left(1 - \xi_i - y_i(x_i^T\beta + \beta_0)\right)$$

$$= \frac{1}{2}\|\beta\|_2^2 - \sum w_i y_i x_i^T \beta - \sum w_i y_i \beta_0 + \sum (C - v_i - w_i)\xi_i + \sum w_i$$

min $\beta$:

$\frac{1}{2}\beta^T\beta - w^T \tilde{X}\beta$

$\Rightarrow$ term $-\frac{1}{2} w^T \tilde{X}^T \tilde{X} \beta$

(minimizer: $\beta = \tilde{X}^T w$)

min $\beta_0$:

$\Rightarrow \sum w_i y_i = 0$

min $\xi_i$:

$C - v_i - w_i = 0$

$$\underbrace{w_i = C - v_i, \; v_i \ge 0}_{w_i \le C}$$