Necessity: if x^* is p.opt, (u^*, v^*) are d.opt, then

$$f^* = g^*$$

$$f(x^*) = g(u^*, v^*)$$

$$= \min_x \; f(x) + \sum u_i^* h_i(x) + \sum v_j^* l_j(x)$$

$$(1) \quad \leq f(x^*) + \sum \underbrace{u_i^*}_{\geq 0} \underbrace{h_i(x^*)}_{\leq 0} + \sum v_j^* \underbrace{l_j(x^*)}_{=0}$$

$$(2) \quad \leq f(x^*)$$

all inequalities must be equality

eq. in (1) implies that x^* minimizes $L(x, u^*, v^*)$ over x

$$\iff 0 \in \partial_x L(x^*, u^*, v^*)$$

$$0 \in \partial f(x^*) + \sum u_i^* \partial h_i(x^*) + \sum v_j^* \partial l_j(x^*)$$

stationarity condition

eq. in (2) implies that $\sum u_i^* h_i(x^*) = 0$

$$\Rightarrow \quad u_i^* h_i(x^*) = 0 \text{ for all } i$$

complementary slackness

primal & dual feasibility of (x^*, u^*, v^*) ✓

Sufficiency: if (x^*, u^*, v^*) satisfy KKT cond, then

$$
\begin{aligned}
g(u^*, v^*) &= \min_x L(x, u^*, v^*) \\
&= L(x^*, u^*, v^*) \quad \text{by stationarity} \\
&= f(x^*) + \underbrace{\sum u_i^* h_i(x^*)}_{\substack{=0 \text{ by} \\ \text{complementary} \\ \text{slackness}}} + \underbrace{\sum v_j^* \ell_j(x^*)}_{=0 \text{ by primal feas.}} \\
&= f(x^*)
\end{aligned}
$$

$\Rightarrow$ x^* is primal optimal and (u^*, v^*) are dual optimal

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if f, h_i, ℓ_j are convex and diff then stationarity cond is $\Longleftrightarrow$

$$0 = \nabla f(x^*) + \sum u_i^* \nabla h_i(x^*) + \sum v_j^* \nabla \ell_j(x^*)$$

for f, h_i, ℓ_j not convex, this is NOT TRUE!
but still diff

$$C = \{x : h_i(x) \le 0 \text{ all } i,\ \ell_j(x) = 0 \text{ all } j\}$$

$$\min \underbrace{f(x) + 1_C(x)}$$

$$\min \quad \tfrac{1}{2} x^T Q x + c^T x$$
$$\text{st} \quad Ax = 0$$

$$L(x, v) = \tfrac{1}{2} x^T Q x + c^T x + v^T A x$$

stationarity: $Qx + c + A^T v = 0$.

comp. slackness: $\emptyset$

p. feas.: $Ax = 0$

d. feas: $\emptyset$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}$$

KKT matrix

$$\min_x \quad -\sum_{i=1}^{n} \log(\alpha_i + x_i) \qquad \text{st.} \quad x \succeq 0, \ \mathbf{1}^T x = 1$$

$$L(x, u, v) = -\sum \log(\alpha_i + x_i) - u^T x + v(\mathbf{1}^T x - 1)$$

stationarity: $\dfrac{-1}{\alpha_i + x_i} - u_i + v = 0 \quad i = 1, \dots, n$

comp slack: $u_i x_i = 0$ all i. $\quad \underbrace{x \succeq 0, \ \mathbf{1}^T x = 1, \ u \succeq 0}_{\text{feas}}$

let $u_i = v - \dfrac{1}{\alpha_i + x_i}$

then $\dfrac{1}{\alpha_i + x_i} \leq v$

$$x_i \left(v - \frac{1}{\alpha_i + x_i} \right) = 0, \quad x \succeq 0, \quad \underline{\mathbf{1}^T x = 1}$$

if $\nu < \frac{1}{\alpha_i}$ then $x_i \neq 0$

so $x_i = 1/\nu - \alpha_i$

if $\nu \geq \frac{1}{\alpha_i}$ then $x_i \not> 0$

so $x_i = 0$

$$x_i = \begin{cases} 1/\nu - \alpha_i & \text{if } \nu < \frac{1}{\alpha_i} \\ 0 & \text{if } \nu \geq \frac{1}{\alpha_i} \end{cases} = \max\{0, 1/\nu - \alpha_i\}$$

KKT reduce to

$$\sum \max\{0, 1/\nu - \alpha_i\} = 1$$

piecewise linear equation in $1/\nu$

SVM

$$\min_{\beta, \beta_0, \xi} \quad \frac{1}{2}\|\beta\|_2^2 + C\sum \xi_i$$

s.t. $\quad \xi_i \geq 0 \qquad\qquad v_i \geq 0$

$\qquad y_i(x_i^T\beta + \beta_0) \geq 1 - \xi_i \qquad w_i \geq 0$

stationarity:

wrt β: $\quad \underline{\beta = \sum w_i y_i x_i}$ ← primal solution in terms of dual solution!

wrt ξ: $\quad C\mathbb{1} - v = w$

wrt β_0: $\quad \sum w_i y_i = 0$

c.s. $\quad v_i \xi_i = 0 \quad , \quad w_i(1 - \xi_i - y_i(x_i^T\beta + \beta_0)) = 0$

support points, i s.t. $w_i > 0$,
completely determine β

$\min \; f(x) \;\text{ st }\; h(x) \leq t$

$0 \ni \partial f(x) + \lambda \partial h(x)$

$\lambda \cdot (h(x) - t) = 0$

take $t = h(x^*)$