stationarity:

$$0 \in \partial f(x) + \sum u_i \partial h_i(x) + \sum v_j \partial \ell_j(x)$$

$\Longleftrightarrow$

x minimizes $L(x,u,v)$

characterizes optimal x in terms of optimal (u,v)

- $\min_x L(x,u,v)$ is non-unique

  still characterizes optimal solution x

- $\min_x L(x,u,v)$ is unique

  determines optimal solution x exactly

$$g(v) = \min_x L(x,v)$$

$$= \min_x \sum f_i(x_i) + v(-a^T x + b)$$

$$= \min_x \left(\sum_i \left(f_i(x_i) - v \cdot a_i x_i \right) + bv \right)$$

$$= -\sum f_i^*(a_i v) + bv$$

$$\max_v g(v) \Longleftrightarrow \min_v \sum f_i^*(a_i v) - bv$$

suppose dual solution v^*

$\Rightarrow$ get primal solution by solving $\nabla f_i(x_i^*) = a_i v$

$$z^T x = \|z\| \left(\frac{z}{\|z\|}\right)^T x$$

$$\leq \|z\| \cdot \max_{\|\omega\| \leq 1} \omega^T x$$

$$= \|z\| \; \|x\|_*$$

$$f^*(y) = \max_x \underbrace{g_x(y)}_{y^T x - f(x)}$$

always convex

$$f^*(y) \geq y^T x - f(x)$$

$$f(x) = \frac{1}{2} x^T Q x, \quad Q \succ 0.$$

$$f^*(y) = \max_x \; y^T x - \frac{1}{2} x^T Q x$$

$$= -\min_x \; \frac{1}{2} x^T Q x - y^T x$$

$$= -\left(\frac{1}{2} y^T Q^{-1} y - y^T Q^{-1} y \right)$$

$$= \frac{1}{2} y^T Q^{-1} y$$

$f(x) = I_C(x)$ indicator

$$f^*(y) = \max_x \; y^T x - I_C(x) = \max_{x \in C} y^T x$$

support function

$f(x) = \|x\|$ norm

$f^*(y) = I\{\|y\|_* \le 1\}$

alternative representation

$f(x) = \max_{\|z\|_* \le 1} z^T x$ support function

already know

$f^*(y) = I\{\|z\|_* \le 1\}$

$\min_\beta \frac{1}{2}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1$

$g = f^*$

$\Longleftrightarrow \min_{z,\beta} \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1$ s.t. $X\beta = z$

$L(z,\beta,u) = \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$

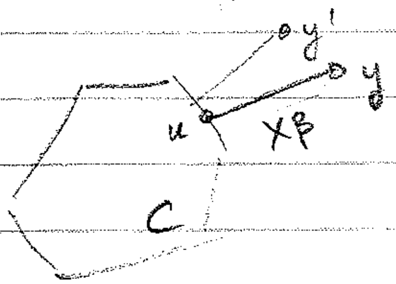
$\min_{z,\beta} L(z,\beta,u) = \min_z \left\{\frac{1}{2}\|y - z\|_2^2 + u^T z\right\} + \min_\beta \left\{\lambda\|\beta\|_1 - (X^T u)^T \beta\right\}$

$= \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2 - \lambda \max_\beta \left\{\left(\frac{X^T u}{\lambda}\right)^T \beta - \|\beta\|_1\right\}$

$g(u) = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2 - I\{\|X^T u\|_\infty \le \lambda\}$

$\max g(u) \Longleftrightarrow \min \frac{1}{2}\|y - u\|_2^2$ s.t. $\|X^T u\|_\infty \le \lambda$

$$\min_u \|y-u\|_2^2 \quad \text{s.t. } u \in C$$



$$C = \{u : \|X^T u\|_\infty \le \lambda\}$$
$$= (X^T)^{-1}\{v : \|v\|_\infty \le \lambda\}$$

inverse image under linear map X^T

Lagrangian: $L(z, \beta, u) = \frac{1}{2}\|y-z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta)$

minimizer of z is $z = y - u$

i.e. $\underline{X\beta = y - u}$

this implies

$X\hat{\beta}(y)$ is Lipschitz cont. in y with $L=1$

$$\|X\hat{\beta}(y) - X\hat{\beta}(y')\|_2 \le \|y-y'\|_2 \quad \text{all } y, y'$$

$$\min_u \; f^*(u) + g^*(-u) \quad \text{dual of}$$

$$\min_x \; f(x) + g(x)$$