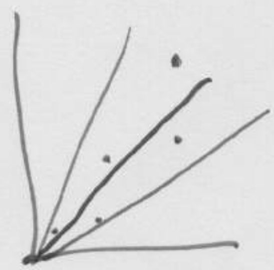
Primal

$$\mathcal{F}^0 \cong \mathbb{R}^n_{++}$$

bijection

$$(x,y,s) \mapsto XS\mathbb{1}$$



$\mathbb{R}^n_{++}$

Recall central path eqtns

$$\left[\begin{array}{rcl} A^T y + s & = & c \\ Ax & = & b \\ XS\mathbb{1} & = & \tau \mathbb{1} \end{array}\right.$$

$$x, s > 0$$

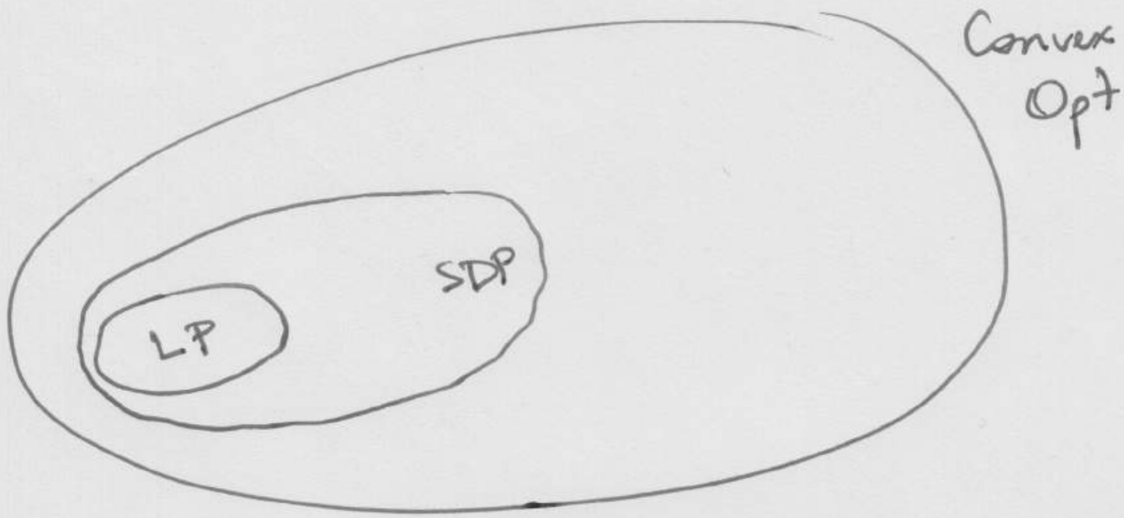
Newton's method:

optimization: $\min f(x)$, $x \mapsto x + \Delta x$

where $[\nabla^2 f(x)] \Delta x = -\nabla f(x)$

solving eqtns: $G(x) = 0$, $x \mapsto x + \Delta x$

$$G'(x) \Delta x = -G(x)$$



Primal SDP

$$\min \; C \cdot X$$
$$\mathcal{A}(X) = b$$
$$X \succeq_c 0$$

$\mathcal{A} : S^n \to \mathbb{R}^m$

$\mathcal{A}^* : \mathbb{R}^m \to S^n$

Typically $\mathcal{A}(X) = \begin{bmatrix} A_1 \cdot X \\ \vdots \\ A_m \cdot X \end{bmatrix}$

$\rightsquigarrow \quad \mathcal{A}^*(y) = \sum_{i=1}^{m} y_i A_i$

$$\min \ 2x_{12} \qquad \iff \qquad \min \ C \cdot X$$

$$\begin{bmatrix} 0 & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq_c 0 \qquad\qquad A_1 \cdot X = b$$

$$X \succeq_c 0$$

for $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = 0$

Dual $\quad \max \ 0 \cdot y$

$$\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \preceq_c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq_c 0 \implies x_{12} = 0$$

so primal opt value is 0.

Dual constraint: $\begin{bmatrix} -y & 1 \\ 1 & 0 \end{bmatrix} \succeq_c 0$

can never happen no matter what y is.

$$\min \quad x_{11} \qquad\qquad \min \quad C \cdot X$$

$$\begin{bmatrix} x_{11} & 1 \\ 1 & x_{22} \end{bmatrix} \succeq_c 0 \iff \begin{array}{l} A_1 \cdot X = b \\ X \succeq_c 0 \end{array}$$

for $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $b = 2$

Dual $\quad \max \quad 2y$

$$\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \preceq_c \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

dual opt value is 0 attained at $y=0$

Primal problem: opt value is 0 but it is <u>not</u> attained

$$\min \quad a\, x_{22}$$

$\leftarrow$ opt val $= 1$ <u>attained</u>

$$\begin{bmatrix} 0 & x_{12} & 1-x_{22} \\ x_{12} & x_{22} & x_{23} \\ 1-x_{22} & x_{23} & x_{33} \end{bmatrix} \succeq_c 0$$

dual: $\quad \max \quad 2y_2$

$\leftarrow$ opt val $= 0$ <u>attained</u>

$$\begin{bmatrix} y_1 & 0 & y_2 \\ 0 & 2y_2 & 0 \\ y_2 & 0 & 0 \end{bmatrix} \preceq_c \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Barrier method for primal SDP:

$$\begin{array}{lll} \min\ C\bullet X & & \min\ C\bullet X - \tau \log\det X \\ \mathcal{A}(X) = b & \rightsquigarrow & \mathcal{A}(X) = b \\ X \succeq_C 0 & & \end{array}$$

The function $F : S^n_{++} \to \mathbb{R}$

$$X \mapsto -\log\det X$$

"log. barrier function for S^n_+"

Observe

$$F(X) = -\log\det X = -\log \prod_{j=1}^{n} \lambda_j(X)$$

$$= -\sum_{j=1}^{n} \log(\lambda_j(X))$$

cf log barrier for $\mathbb{R}^n_+$: $-\sum \log x_j$

Recall $f(t) = -\log t \rightsquigarrow f'(t) = -\frac{1}{t}$

For $F(X) = -\log\det X \rightsquigarrow \nabla F(X) = -X^{-1}$

$d_F(X,S)$ = measure of proximity of XS to μI

A "feasible" method would satisfy

$$\underbrace{\begin{aligned} \mathcal{A}^*(y) + S - C &= 0 \\ \mathcal{A}(X) - b &= 0 \\ XS &\neq \tau I \end{aligned}}$$

$$G(X, y, S) - \begin{pmatrix} 0 \\ 0 \\ \tau I \end{pmatrix} = 0.$$

"natural" linearization of $XS - \tau I = 0$

$$\rightsquigarrow \quad X \Delta S + S \Delta X = \tau I - XS$$

Neat calculus fact: $F(X) = -\log\det X$

$$\nabla F(X) = -X^{-1}$$

$$\nabla^2 F(X)[Z] = X^{-1} Z X^{-1}$$

Recall Lovasz theta function

$G = (V, E)$

$$\theta(G) = \max \ \mathbb{1}\mathbb{1}^T \bullet X$$
$$I \bullet X = 1$$
$$X_{ij} = 0 \quad ij \in E$$
$$X \succeq 0$$