Appendix A
Properties of positive (semi)definite matrices

In this appendix we list some well-known properties of positive (semi)definite matrices which are used in this monograph. The proofs which are omitted here may be found in [85]. A more detailed review of the matrix analysis which is relevant for SDP is given by Jarre in [94].

A.1 CHARACTERIZATIONS OF POSITIVE (SEMI)DEFINITENESS

Theorem A.1 *Let $X \in \mathcal{S}_n$. The following statements are equivalent:*

- $X \in \mathcal{S}_n^+$ *or* $X \succeq 0$ *(X is positive semidefinite);*
- $z^T X z \geq 0 \quad \forall z \in \mathbf{R}^n$*;*
- $\lambda_{\min}(X) \geq 0$*;*
- *All principal minors of X are nonnegative;*
- $X = LL^T$ *for some* $L \in \mathbf{R}^{n \times n}$.

We can replace 'positive semidefinite' by 'positive definite' in the statement of the theorem by changing the respective nonnegativity requirements to positivity, and by requiring that the matrix L in the last item be nonsingular. If X is positive definite ($X \in \mathcal{S}_n^{++}$ or $X \succ 0$), the matrix L can be chosen to be lower triangular, in which case we call $X = LL^T$ the *Choleski factorization of X.*

NB: In this monograph positive (semi)definite matrices are necessarily symmetric, *i.e.* we will use 'positive (semi)definite' instead of 'symmetric positive (semi)definite'.[1]

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[1] In the literature a matrix $X \in \mathbf{R}^{n \times n}$ is sometimes called positive (semi)definite if its symmetric part $\frac{1}{2}(X + X^T)$ is positive (semi)definite.

As an immediate consequence of the second item in Theorem A.1 we have that

$$X \in \mathcal{S}_n^+ \Leftrightarrow AXA^T \in \mathcal{S}_n^+$$

for any given, nonsingular $A \in \mathbf{R}^{n \times n}$.

Another implication is that a block diagonal matrix is positive (semi)definite if and only if each of its diagonal blocks is positive (semi)definite.

A.2 SPECTRAL PROPERTIES

The characterization of positive (semi)definiteness in terms of nonnegative eigenvalues follows from the Raleigh–Ritz theorem.

Theorem A.2 (Raleigh–Ritz) *Let $A \in \mathcal{S}_n$. Then*

$$\lambda_{\min}(A) = \min_{z \in \mathbf{R}^n} \left\{ z^T A z \mid \|z\| = 1 \right\}. \tag{A.1}$$

It is well known that a symmetric matrix has an orthonormal set of eigenvectors, which implies the following result.

Theorem A.3 (Spectral decomposition) *Let $A \in \mathcal{S}_n$. Now A can be decomposed as*

$$A = Q^T \Lambda Q = \sum_{i=1}^{n} \lambda_i(A) q_i q_i^T$$

where Λ is a diagonal matrix with the eigenvalues $\lambda_i(A)$ $(i = 1, \ldots, n)$ of A on the diagonal, and Q is an orthogonal matrix with a corresponding set of orthonormal eigenvectors $q_1, \ldots, q_n$ of A as columns.

Since $\lambda_i(X) \geq 0$ $(i = 1, \ldots, n)$ if $X \in \mathcal{S}_n^+$, we can define the *symmetric square root factorization* of $X \in \mathcal{S}_n^+$:

$$X^{\frac{1}{2}} = \sum_{i=1}^{n} \sqrt{\lambda_i(X)} q_i q_i^T \quad \left(X \in \mathcal{S}_n^+\right).$$

Note that $X^{\frac{1}{2}} X^{\frac{1}{2}} = X$; $X^{\frac{1}{2}}$ is the only matrix with this property.

Theorem A.4 *Let $X \in \mathcal{S}_n^{++}$ and $S \in \mathcal{S}_n^{++}$. Then all the eigenvalues of XS are real and positive.*

Proof:

The proof is immediate by noting that

$$XS \sim \left(X^{\frac{1}{2}}\right)^{-1} XS\left(X^{\frac{1}{2}}\right) = X^{\frac{1}{2}}SX^{\frac{1}{2}} \succ 0.$$

□

We will often use the notation $X^{-\frac{1}{2}} := \left(X^{\frac{1}{2}}\right)^{-1}$.

The eigenvalues of a symmetric matrix can be viewed as smooth functions on $\mathcal{S}_n$ in a sense made precise by the following theorem.

Theorem A.5 (Rellich) *Let an interval $(a, b) \subset \mathbf{R}$ be given. If $A : (a,b) \mapsto \mathcal{S}_n$ is a continuously differentiable function, then there exist n continuously differentiable functions $\lambda_i : (a,b) \mapsto \mathbf{R}$ $(i = 1, \ldots, n)$ such that $\lambda_1(t), \ldots, \lambda_n(t)$ give the values of the eigenvalues of $A(t)$ for each $t \in (a, b)$.*

The next lemma shows what happens to the spectrum of a positive semidefinite matrix if a skew symmetric matrix is added to it, in the case where the eigenvalues of the sum of the two matrices remain real numbers.

Lemma A.1 *Let $Q \in \mathcal{S}_n^{++}$,and let $M \in \mathbf{R}^{n\times n}$ be skew–symmetric ($M = -M^T$). One has $\det(Q + M) > 0$. Moreover, if $\lambda_i(Q + M) \in \mathbf{R}$ $(i = 1, \ldots, n)$, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q + M) \leq \lambda_{\max}(Q + M) \leq \lambda_{\max}(Q).$$

Proof:

First note that $Q + M$ is nonsingular since for all nonzero $x \in \mathbf{R}^n$:

$$x^T(Q + M)x = x^TQx > 0,$$

using the skew symmetry of M. We therefore know that

$$\psi(t) := \det[Q + tM] \neq 0 \quad \forall t \in \mathbf{R},$$

since tM remains skew–symmetric. One now has that ψ is a continuous function of t which is nowhere zero and strictly positive for $t = 0$ as $\det(Q) > 0$. This shows $\det(Q + M) > 0$.

To prove the second part of the lemma, assume $\lambda > 0$ is such that $\lambda > \lambda_{\max}(Q)$. It then follows that $Q - \lambda I \prec 0$. By the same argument as above we then have $(Q + M) - \lambda I$ nonsingular, or

$$\det\left((Q + M) - \lambda I\right) \neq 0.$$

This implies that λ cannot be an eigenvalue of $Q + M$. Similarly, $Q + M$ cannot have an eigenvalue smaller than $\lambda_{\min}(Q)$. This gives the required result. □

The *spectral norm* $\|\cdot\|_2$ on $\mathbb{R}^{n\times n}$ is the norm induced by the Euclidean vector norm:

$$\|A\|_2 := \max_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \quad (A \in \mathbb{R}^{n\times n}).$$

By the Raleigh–Ritz theorem, the spectral norm and spectral radius $\rho(\cdot)$ coincide for symmetric matrices:

$$\|A\|_2 = \rho(A) := \max_i |\lambda_i(A)| \quad \forall A \in \mathcal{S}_n.$$

The location of the eigenvalues of a matrix is bounded by the famous Gerschgorin theorem. For symmetric matrices the theorem states that

$$\lambda_k(A) \in \bigcup_{i=1}^{n} \left\{ t \;\middle|\; |t - a_{ii}| \le \sum_{j\neq i} |a_{ij}| \right\}, \quad k = 1,\ldots,n, \quad A \in \mathcal{S}_n.$$

As a consequence we find that the so–called diagonally dominant matrices are positive semi-definite.

Theorem A.6 (Diagonally dominant matrix is PSD) *A matrix $A \in \mathcal{S}_n$ is called diagonally dominant if*

$$a_{ii} \ge \sum_{j\neq i} |a_{ij}|, \quad i = 1,\ldots,n.$$

If A is diagonally dominant, then $A \in \mathcal{S}_n^+$.

A.3 THE TRACE OPERATOR AND THE FROBENIUS NORM

The trace of an $n \times n$ matrix *A* is defined as

$$\mathbf{Tr}\,(A) = \sum_{i=1}^{n} a_{ii}.$$

The trace is clearly a linear operator and has the following properties.

Theorem A.7 *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. Then the following holds:*

- $\mathbf{Tr}\,(A) = \sum_{i=1}^{n} \lambda_i(A)$;
- $\mathbf{Tr}\,(A) = \mathbf{Tr}\,(A^T)$;
- $\mathbf{Tr}\,(AB) = \mathbf{Tr}\,(BA)$
- $\mathbf{Tr}\,(AB^T) = \mathbf{vec}(A)^T\mathbf{vec}(B) = \sum_{i,j=1}^{n} a_{ij}b_{ij}$.

The last item shows that we can view the usual Euclidean inner product on $\mathbb{R}^{n^2}$ as an inner product on $\mathbb{R}^{n\times n}$:

$$\langle A, B\rangle := \mathbf{Tr}\,(AB^T) = \mathbf{Tr}\,(B^TA) = \mathbf{Tr}\,(A^TB) = \mathbf{Tr}\,(B^TA)\,. \quad \text{(A.2)}$$

The inner product in (A.2) induces the so-called *Frobenius* (or Euclidean) norm on $\mathbf{R}^{n\times n}$:

$$\|A\|^2 := \langle A, A\rangle = \mathbf{Tr}\left(AA^T\right) = \sum_{i,j=1}^{n} a_{ij}^2.$$

It now follows from the first item in Theorem (A.7) that

$$\|C\|^2 = \sum_{i=1}^{n} \lambda_i(C)^2 \text{ if } C \in \mathcal{S}_n.$$

The Frobenius and spectral norms are sub-multiplicative, *i.e.*

$$\|AB\| \leq \|A\|\|B\|,\ \|AB\|_2 \leq \|A\|_2\|B\|_2 \quad \forall A \in \mathbf{R}^{n\times n},\ B \in \mathbf{R}^{n\times n},$$

and $\|A\|_2 \leq \|A\|$ for all $A \in \mathbf{R}^{n\times n}$.

One can easily prove the useful inequality:

$$\mathbf{Tr}\,(AB) \leq \lambda_{\max}(A)\mathbf{Tr}\,(B), \quad \text{for } A, B \succeq 0, \tag{A.3}$$

that is equivalent to

$$\|AB\| \leq \|A\|_2\|B\|, \quad \text{for } A, B \in \mathbf{R}^{n\times n}. \tag{A.4}$$

Inequality (A.4) follows from (A.3) by replacing *A* by AA^T and *B* by BB^T in (A.3). Conversely, (A.3) follows from (A.4) by replacing *A* by $A^{\frac{1}{2}}$ and *B* by $B^{\frac{1}{2}}$ in (A.4).

Theorem A.8 (Fejer) *A matrix $A \in \mathcal{S}_n$ is positive semidefinite if and only if $\langle A, B\rangle \geq 0$ for all $B \in \mathcal{S}_n^+$. In other words, the cone $\mathcal{S}_n^+$ is self-dual.*

Proof:

Let $A \in \mathcal{S}_n^+$ and $B \in \mathcal{S}_n^+$; then

$$\langle A, B\rangle = \mathbf{Tr}\left(A^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}\right) = \mathbf{Tr}\left(A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) = \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2 \geq 0.$$

Conversely, if $A \in \mathcal{S}_n$ and $\langle A, B\rangle \geq 0$ for all $B \in \mathcal{S}_n^+$, then let $x \in \mathbf{R}^n$ be given and set $B = xx^T \in \mathcal{S}_n^+$. Now

$$0 \leq \langle A, B\rangle = \mathbf{Tr}\left(Axx^T\right) = \sum_{i,j=1}^{n} a_{ij}x_ix_j = x^TAx.$$

$\square$

For positive semidefinite matrices, the trace dominates the Frobenius norm, *i.e.*

$$\mathbf{Tr}\,(X) \geq \|X\| \quad \forall X \in \mathcal{S}_n^+.$$

This follows by applying the inequality

$$\sum_{i=1}^{n} x_i \geq \sqrt{\sum_{i=1}^{n} x_i^2} \quad \forall x \in \mathbf{R}_n^+$$

to the nonnegative eigenvalues of X. Similarly, one can apply the arithmetic-geometric mean inequality

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i \quad \forall x \in \mathbf{R}_n^+$$

to the eigenvalues of $X \in \mathcal{S}_n^+$ to obtain the inequality

$$\frac{1}{n}\mathbf{Tr}\,(X) \geq (\det(X))^{1/n} \quad \forall X \in \mathcal{S}_n^+,$$

where we have used the fact that $\det(A) = \prod_i \lambda_i(A)$ for any $A \in \mathbf{R}^{n\times n}$.

Lemma A.2 *If $X \in \mathcal{S}_n^+$ and $S \in \mathcal{S}_n^+$ and $\mathbf{Tr}\,(XS) = 0$, then $XS = SX = 0$.*

Proof:

By the properties of the trace operator

$$\mathbf{Tr}\,(XS) = \mathbf{Tr}\,\left(X^{\frac{1}{2}}X^{\frac{1}{2}}S^{\frac{1}{2}}S^{\frac{1}{2}}\right) = \mathbf{Tr}\,\left(S^{\frac{1}{2}}X^{\frac{1}{2}}X^{\frac{1}{2}}S^{\frac{1}{2}}\right) = \left\|S^{\frac{1}{2}}X^{\frac{1}{2}}\right\|^2.$$

Thus if $\mathbf{Tr}\,(XS) = 0$, it follows that $S^{\frac{1}{2}}X^{\frac{1}{2}} = 0$. Pre-multiplying by $S^{\frac{1}{2}}$ and post-multiplyingby $X^{\frac{1}{2}}$ yields $SX = 0$, which in turn implies $(SX)^T = XS = 0$. □

The following lemma is used to prove that the search directions for the interior point methods described in this monograph are well defined. The proof given here is based on a proof given by Faybusovich [54].

Lemma A.3 *Let $A_i \in \mathcal{S}_n$ $(i = 1, \ldots, m)$ be linearly independent, and let $0 \neq Y \in \mathcal{S}_n^+$, $Z \in \mathcal{S}_n^{++}$. The matrix $M \in \mathcal{S}_m$ with entries*

$$m_{ij} := \mathbf{Tr}\,(A_i Z A_j Y), \quad i,j = 1, \ldots, m$$

is positive definite.

Proof:

We prove that the quadratic form

$$q(x) = x^T M x = \sum_{i,j=1}^{m} m_{ij}x_i x_j$$

is strictly positive for all nonzero $x \in \mathbf{R}^n$. To this end, note that for given $x \neq 0$,

$$q(x) = \mathbf{Tr}\left(\left(\sum_{i=1}^{m} x_i A_i\right) Z \left(\sum_{j=1}^{m} x_j A_j\right) Y\right).$$

Denoting $A(x) := \sum_{i=1}^{m} x_i A_i$ (which is nonzero by the linear independence of the $A_i's$), one has:

$$q(x) = \mathbf{Tr}\,(A(x)ZA(x)Y) > 0,$$

where the inequality follows from $0 \neq A(x)ZA(x) \succeq 0$ and $Y \succ 0$. □

A.4 THE LÖWNER PARTIAL ORDER AND THE SCHUR COMPLEMENT THEOREM

We define a partial ordering on $\mathcal{S}_n$ via:

$$A \succeq B \iff A - B \in \mathcal{S}_n^+.$$

This partial ordering is called the *Löwner partial order* on $\mathcal{S}^n$. (It motivates the alternative notation $X \succeq 0$ instead of $X \in \mathcal{S}_n^+$.) It follows immediately that

$$A \succeq B \iff C^T A C \succeq C^T B C \quad \forall C \in \mathbf{R}^{n \times n}.$$

One also has

$$A \succeq B \iff B^{-1} \succeq A^{-1} \quad (A \in \mathcal{S}_N^{++},\ B \in \mathcal{S}_n^{++}).$$

The Schur complement theorem gives us useful ways to express positive semidefiniteness of matrices with a block structure.

Theorem A.9 (Schur complement) *If*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where A is positive definite and C is symmetric, then the matrix

$$C - B^T A^{-1} B$$

is called the Schur complement of A in X. The following are equivalent:

- *M is positive (semi)definite;*
- *$C - B^T A^{-1} B$ is positive (semi)definite.*

Proof:

The result follows by setting $D = -A^{-1}B$, and noting that

$$\begin{bmatrix} I & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & D \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}.$$

Since a block diagonal matrix is positive (semi)definite if and only if its diagonal blocks are positive (semi)definite, the proof is complete. □

Appendix B
Background material on convex optimization

In this Appendix we give some background material on convex analysis, convex optimization and nonlinear programming. All proofs are omitted here but may be found in the books by Rockafellar [160] (convex analysis) and Bazaraa *et al.* [16] (nonlinear programming).

B.1 CONVEX ANALYSIS

Definition B.1 (Convex set) *Let two points $x^1, x^2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$ be given. Then the point*

$$x = \lambda x^1 + (1 - \lambda)x^2$$

is a convex combination *of the two points x^1, x^2.*

The set $\mathcal{C} \subset \mathbf{R}^n$ is called convex, *if all convex combinations of any two points $x^1, x^2 \in \mathcal{C}$ are again in $\mathcal{C}$.*

Definition B.2 (Convex function) *A function $f : \mathcal{C} \to R$ defined on a convex set $\mathcal{C}$ is called* convex *if for all $x^1, x^2 \in \mathcal{C}$ and $0 \leq \lambda \leq 1$ one has*

$$f(\lambda x^1 + (1 - \lambda)x^2) \leq \lambda f(x^1) + (1 - \lambda)f(x^2).$$

The function is called strictly convex *if the last inequality is strict.*

A function is convex if and only if its epigraph is convex.

Definition B.3 (Epigraph) *The epigraph of a function $f : \mathcal{C} \to R$ is the $(n + 1)$-dimensional set*

$$\{(x, \tau) \ : \ f(x) \leq \tau, x \in \mathcal{C}, \ \tau \in \mathbf{R}\}.$$

Theorem B.1 *A twice differentiable function f is convex (resp. strictly convex) on an open set $\mathcal{C}$ if and only if its Hessian $\nabla^2 f$ is positive semidefinite (resp. positive definite) on $\mathcal{C}$.*

Example B.1 *The function $f : \mathcal{S}_n^{++} \mapsto \mathbf{R}$ defined by $f(X) = -\log(\det(X))$ has a positive definite Hessian and is therefore strictly convex. (This is proven in Appendix C.)* □

Strictly convex functions are useful for proving 'uniqueness properties', due to the following result.

Theorem B.2 *If a strictly convex function has a minimizer over a convex set, then this minimizer is unique.*

Definition B.4 (Convex cone) *The set $\mathcal{K} \subset \mathbf{R}^n$ is a* convex cone *if it is a convex set and for all $x \in \mathcal{K}$ and $0 < \lambda$ one has $\lambda x \in \mathcal{K}$.*

Example B.2 *Four examples of convex cones in $\mathcal{S}_n$ are:*

- *the symmetric positive semidefinite cone:*

$$\mathcal{S}_n^+ := \left\{ A \in \mathcal{S}_n \;\middle|\; x^T X x \geq 0 \quad \textit{for all } x \in \mathbf{R}^n \right\};$$

- *the copositive cone:*

$$\mathcal{C}_n := \left\{ A \in \mathcal{S}_n \;\middle|\; x^T A x \geq 0 \quad \textit{for all } x \in \mathbf{R}_+^n \right\};$$

- *the cone of completely positive matrices:*

$$\mathcal{C}_n^* = \left\{ A \in \mathcal{S}_n \;\middle|\; A = \sum_{i=1}^{k} x_i x_i^T,\; x_i \in \mathbf{R}_+^n \;\; (i = 1, \ldots, k) \;\; \textit{for any } k \right\};$$

- *the cone of nonnegative matrices:*

$$\mathcal{N}_n = \{ A \in \mathcal{S}_n \mid a_{ij} \geq 0 \;\; (i, j = 1, \ldots, n) \}.$$

□

Definition B.5 (Face (of a cone)) *A subset $\mathcal{F}$ of a convex cone $\mathcal{K}$ is called a face of $\mathcal{K}$ if for all $x \in \mathcal{F}$ and $y, z \in \mathcal{K}$ one has $x = y + z$ if and only if $y \in \mathcal{F}$ and $z \in \mathcal{F}$.*

Example B.3 *An example of a face of the cone of positive semidefinite matrices* $\mathcal{S}_n^+$ *is*

$$\mathcal{F} := \left\{ \left[\begin{array}{cc} U & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{array} \right] \;\middle|\; U \in \mathcal{S}_r^+ \right\}.$$

Note that if $A \in \mathcal{S}_n^+$ *and* $B \in \mathcal{S}_n^+$, *then* $A + B \in \mathcal{F}$ *if and only if* $A \in \mathcal{F}$ *and* $B \in \mathcal{F}$. □

Definition B.6 (Extreme ray (of a cone)) *A subset of a convex cone* $\mathcal{K}$ *is called an* extreme ray *of* $\mathcal{K}$ *if it is a one dimensional face of* $\mathcal{K}$, i.e. *a face that is a half line emanating from the origin.*

Example B.4 *Any* $0 \neq x \in \mathbf{R}^n$ *defines an extreme ray of the cone of positive semidefinite matrices* $\mathcal{S}_n^+$ *via*

$$\left\{ U \mid U = cxx^T,\ c > 0 \right\}.$$

Similarly, any $y \in \mathbf{R}_+^n$ *defines an extreme ray of the cone of completely positive semidefinite matrices* $\mathcal{C}_n^*$ *via*

$$\left\{ U \mid U = cyy^T,\ c > 0 \right\}.$$

□

Definition B.7 *Let a convex set* $\mathcal{C}$ *be given. The point* $x \in \mathcal{C}$ *is in the* relative interior *of* $\mathcal{C}$ *if for all* $\hat{x} \in \mathcal{C}$ *there exists* $\tilde{x} \in \mathcal{C}$ *and* $0 < \lambda < 1$ *such that* $x = \lambda\hat{x} + (1-\lambda)\tilde{x}$. *The set of relative interior points of the set* $\mathcal{C}$ *will be denoted by* $\mathrm{ri}\,(\mathcal{C})$.

Theorem B.3 *Assume that* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are nonempty convex sets and* $\mathrm{ri}\,(\mathcal{C}_1) \cap \mathrm{ri}\,(\mathcal{C}_2) \neq \emptyset$. *Then*

$$\mathrm{ri}\,(\mathcal{C}_1 \cap \mathcal{C}_2) = \mathrm{ri}\,(\mathcal{C}_1) \cap \mathrm{ri}\,(\mathcal{C}_2).$$

Example B.5 *Note that the interior of the cone of positive semidefinite matrices* $\mathcal{S}_n^+$ *is the cone of positive definite matrices* $\mathcal{S}_n^{++}$. *Let* $\mathcal{A} \subset \mathcal{S}_n$ *denote an affine space. Assume that there exists an* $X \in \mathcal{A}$ *that is also positive definite. Then, by the last theorem,*

$$\mathrm{ri}\,\left(\mathcal{A} \cap \mathcal{S}_n^+\right) = \mathrm{ri}\,(\mathcal{A}) \cap \mathrm{ri}\,\left(\mathcal{S}_n^+\right) = \mathcal{A} \cap \mathcal{S}_n^{++},$$

since $\mathrm{ri}\,(\mathcal{A}) = \mathcal{A}$. □

Theorem B.4 (Separation theorem for convex sets) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be nonempty convex sets in $\mathbf{R}^k$. There exists a $r \in \mathbf{R}^k$ such that*

$$\sup_{x \in \mathcal{C}_1} r^T x \leq \inf_{y \in \mathcal{C}_2} r^T y$$

and

$$\inf_{x \in \mathcal{C}_1} r^T x < \sup_{y \in \mathcal{C}_2} r^T y$$

if and only if the relative interiors of $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint.

The second inequality merely excludes the uninteresting case where the separating hyperplane contains both $\mathcal{C}_1$ and $\mathcal{C}_2$, *i.e.* it ensures so-called *proper separation.*

B.2 DUALITY IN CONVEX OPTIMIZATION

We consider the generic convex optimization problem:

$$\begin{array}{lll} (CO) & p^* = \inf_x & f(x) \\ & \text{s.t.} & g_j(x) \leq 0, \quad j = 1, \cdots, m \\ & & x \in \mathcal{C}, \end{array} \tag{B.1}$$

where $\mathcal{C} \subseteq \mathbf{R}^n$ is a convex set and $f, g_1, \cdots, g_m$ are differentiable convex functions on $\mathcal{C}$ (or on an open set that contains the set $\mathcal{C}$).

For the convex optimization problem (CO) one defines the Lagrange function (or Lagrangian)

$$L(x, y) := f(x) + \sum_{j=1}^{m} y_j g_j(x) \tag{B.2}$$

where $x \in \mathcal{C}$ and $y \geq 0$.

Definition B.8 *A vector pair $(\bar{x}, \bar{y}) \in \mathbf{R}^{n+m}$, $\bar{x} \in \mathcal{C}$ and $\bar{y} \geq 0$ is called a saddle point of the Lagrange function L if*

$$L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y}) \tag{B.3}$$

for all $x \in \mathcal{C}$ and $y \geq 0$.

One easily sees that (B.3) is equivalent with

$$L(\bar{x}, y) \leq L(x, \bar{y}) \quad \text{for all} \quad x \in \mathcal{C}, \quad y \geq 0.$$

Lemma B.1 *The vector $(\bar{x}, \bar{y}) \in \mathbf{R}^{n+m}$, $\bar{x} \in \mathcal{C}$ and $\bar{y} \geq 0$ is a saddle point of $L(x, y)$ if and only if*

$$\inf_{x \in \mathcal{C}} \sup_{y \geq 0} L(x, y) = L(\bar{x}, \bar{y}) = \sup_{y \geq 0} \inf_{x \in \mathcal{C}} L(x, y). \tag{B.4}$$

Since we can reformulate (CO) as

$$p^* = \inf_{x \in C} \sup_{y \geq 0} \left\{ f(x) + \sum_{j=1}^{m} y_j g_j(x) \right\},$$

it follows that $\bar{x}$ is an optimal solution of problem (CO) if there exists a $\bar{y} \geq 0$ such that $(\bar{x}, \bar{y})$ is a saddle point of the Lagrangian.

To ensure the existence of a saddle point of the Lagrangian, it is sufficient to require the so-called Slater regularity condition (Slater constraint qualification).

Assumption B.1 (Slater regularity) *There exists an $x \in \mathrm{ri}\,(C)$ such that*

- $g_j(x) < 0$ *if g_j not linear or affine;*
- $g_j(x) \leq 0$ *if g_j is linear or affine.*

Under the Slater regularity assumption, we therefore have a one-to-one correspondence between a saddle point of the Lagrangian and an optimal solution of (CO).

Theorem B.5 (Karush–Kuhn–Tucker) *The convex optimization problem (CO) is given. Assume that the Slater regularity condition is satisfied. The vector $\bar{x}$ is an optimal solution of (CO) if and only if there is a vector $\bar{y}$ such that $(\bar{x}, \bar{y})$ is a saddle point of the Lagrange function L.*

The formulation of the saddle point condition in Lemma (B.1) motivates the concept of the Lagrangian dual.

Definition B.9 (Lagrangian dual) *Denote*

$$\psi(y) = \inf_{x \in C} \{ f(x) + \sum_{j=1}^{m} y_j g_j(x) \}.$$

The problem

$$\psi(y)$$
$$y \geq 0$$

is called the Lagrangian dual of the convex optimization problem (CO).

It is straightforward to show the so-called weak duality property.

Theorem B.6 (Weak duality) *If $\bar{x}$ is a feasible solution of (CO) and $\bar{y} \geq 0$, then*

$$\psi(\bar{y}) \leq f(\bar{x})$$

and equality holds if and only if $\inf_{x \in \mathcal{C}} \{f(x) + \sum_{j=1}^{m} \bar{y}_j g_j(x)\} = f(\bar{x})$.

Under the Slater regularity assumption we have a stronger duality result, by the Karush–Kuhn–Tucker theorem and Lemma (B.1).

Theorem B.7 (Strong duality) *Assume that (CO) satisfies the Slater regularity condition, and let* $\bar{x}$ *be a feasible solution of (CO). Now the vector* $\bar{x}$ *is an optimal solution of (CO) if and only if there exists a* $\bar{y} \geq 0$ *such that* $\bar{y}$ *is an optimal solution of the Lagrangian dual problem and*

$$\psi(\bar{y}) = f(\bar{x}).$$

B.3 THE KKT OPTIMALITY CONDITIONS

We now state the Karush–Kuhn–Tucker (KKT) necessary and sufficient optimality conditions for problem (CO). First we define the notion of a KKT point.

Definition B.10 (KKT point) *The vector* $(\hat{x}, \bar{y}) \in \mathcal{C} \times \mathbf{R}^m$ *is called a Karush–Kuhn–Tucker (KKT) point of (CO) if*

$$\begin{array}{ll} (i) & g_j(\hat{x}) \leq 0, \text{ for all } j = 1, \ldots, m \\ (ii) & 0 = \nabla f(\hat{x}) + \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\hat{x}) \\ (iii) & \sum_{j=1}^{m} \bar{y}_j g_j(\hat{x}) = 0 \\ (iv) & \bar{y} \geq 0. \end{array}$$

A KKT point is a saddle point of the Lagrangian L of (CO). Conversely, a saddle point of L, is a KKT point of (CO). This leads us to the following result.

Theorem B.8 (KKT conditions) *If* $(\bar{x}, \bar{y})$ *is a KKT point, then* $\bar{x}$ *is an optimal solution of (CO). Conversely — under the Slater regularity assumption — a feasible solution* $\bar{x}$ *of (CO) is optimal if there exists a* $\bar{y} \in \mathbf{R}^m$ *such that* $(\bar{x}, \bar{y})$ *is a KKT point.*

We say that $x \in \mathcal{C}$ meets the KKT conditions if there exists a $y \geq 0$ such that (x, y) is a KKT point of (CO).

If we drop the convexity requirements on f and g_i in the statement of (CO), then the KKT conditions remain *necessary* optimality conditions under the Slater regularity assumption.

Appendix C
The function log det(X)

In this appendix we develop the matrix calculus needed to derive the gradient and Hessian of the function log $\det(X)$, and show that it is a strictly concave function.

Lemma C.1 *Let* $f : \mathrm{int}\,(\mathcal{S}_n^+) \mapsto \mathbf{R}$ *be given by*

$$f(X) = \log\det X,$$

Denoting

$$\nabla f(X) := \begin{bmatrix} \frac{\partial f(X)}{\partial x_{11}} & \cdots & \frac{\partial f(X)}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(X)}{\partial x_{n1}} & \cdots & \frac{\partial f(X)}{\partial x_{nn}} \end{bmatrix},$$

one has $\nabla f(X) = X^{-1}$.

Proof:

Let $X \in \mathrm{int}\,(\mathcal{S}_n^+)$ be given and let $H \in \mathcal{S}_n$ be such that $X + H \in \mathrm{int}(\mathcal{S}_n^+)$. One has

$$\begin{aligned} f(X+H) - f(X) &= \log\det(X+H) - \log\det(X) \\ &= \log\det\left(X^{-1}(X+H)\right) \\ &= \log\det\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right). \end{aligned}$$

By the arithmetic-geometric inequality applied to the eigenvalues of $X^{-\frac{1}{2}}HX^{-\frac{1}{2}}$ one has

$$\begin{aligned} \log\det\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right) &\leq \log\left(\frac{1}{n}\mathbf{Tr}\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\right)^n \\ &= n\log\left(\frac{1}{n}\mathbf{Tr}\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\right) \end{aligned}$$

$$= n \log \left(1 + \frac{1}{n} \mathbf{Tr} \left(X^{-\frac{1}{2}} H X^{-\frac{1}{2}} \right) \right).$$

Using the well-known inequality $\log(1+t) \le t$ we arrive at

$$f(X+H) - f(X) \le \mathbf{Tr} \left(X^{-\frac{1}{2}} H X^{-\frac{1}{2}} \right) = \langle X^{-1}, H \rangle.$$

This shows that X^{-1} is a subgradient of f at X. Since f is assumed differentiable, the subgradient is unique and equals the gradient $\nabla f(X)$. □

The proof of the next result is trivial.

Lemma C.2 *Let $f : \operatorname{int}(\mathcal{S}_n^+) \mapsto \mathbf{R}$ be given by*

$$f(X) = \mathbf{Tr}(CX),$$

where $C \in \mathcal{S}_n$. One has $\nabla f(X) = C$.

The following result is used to derive the Hessian of the log-barrier function $f_{bar}(X) = -\log\det(X)$.

Lemma C.3 *Let $f : \mathcal{S}_n^{++} \mapsto \mathbf{R}$ be given by*

$$f(X) = \log\det X.$$

If $\nabla^2 f$ denotes the derivative of $\nabla f : X \mapsto X^{-1}$ with respect to X, then $\nabla^2 f(X)$ is the linear operator which satisfies

$$\nabla^2 f(X)H = -X^{-1}HX^{-1}, \quad \forall H \in \mathcal{S}_n,$$

for a given invertible X.

Proof:

Let $\mathrm{L}(\mathcal{S}_n, \mathcal{S}_n)$ denote the space of linear operators which map $\mathcal{S}_n$ to $\mathcal{S}_n$. The Frechet derivative of ∇f is defined as the (unique) function $\nabla^2 f : \mathcal{S}_n \mapsto \mathrm{L}(\mathcal{S}_n, \mathcal{S}_n)$ such that

$$\lim_{\|H\| \to 0} \frac{\left\| \nabla f(X+H) - \nabla f(X) - \nabla^2 f(X)H \right\|}{\|H\|} = 0. \tag{C.1}$$

We show that $\nabla^2 f(X)H := -X^{-1}HX^{-1}$ satisfies (C.1). To this end, let $H \in \mathcal{S}_n$ be such that $(X+H)$ is invertible, and consider

$$\begin{aligned}
& \left\| \nabla f(X+H) - \nabla f(X) - \nabla^2 f(X)H \right\| \\
= & \left\| (X+H)^{-1} - X^{-1} + X^{-1}HX^{-1} \right\| \\
= & \left\| (X+H)^{-1} \left(I - (X+H)X^{-1} + (X+H)X^{-1}HX^{-1} \right) \right\| \\
= & \left\| (X+H)^{-1} \left(HX^{-1}HX^{-1} \right) \right\| \\
\le & \left\| (X+H)^{-1} \right\| \|H\| \left\| X^{-1}HX^{-1} \right\|,
\end{aligned}$$

which shows that (C.1) indeed holds. □

By Lemma A.3, the Hessian of the function $f(X) = -\log\det(X)$ is a positive definite operator which implies that f is strictly convex on $\mathcal{S}_n^{++}$. We state this observation as a theorem.

Theorem C.1 *The function $f : \mathcal{S}_n^{++} \mapsto \mathbf{R}$ defined by*

$$f(X) = -\log\det(X)$$

is strictly convex.

An alternative proof of this theorem is given in [85] (Theorem 7.6.6).

Appendix D
Real analytic functions

This appendix gives some elementary properties of analytic functions which are used in this monograph. It is based on notes by M. Halická [76].

Definition D.1 *A function $f(x) : \mathbf{R} \to \mathbf{R}$ is said to be analytic at $x = a$ if there exist $r > 0$ and $\{a_n\}$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \quad \forall x : |x-a| < r. \tag{D.1}$$

Remark D.1 *Taking the nth derivative in (D.1), it is easy to see that $a_n = \frac{f^{(n)}(a)}{n!}$. Hence the series in (D. 1) is the Taylor series of $f(x)$ at $x = a$.*

Remark D.2 *A function $f(x)$ that is analytic at $x = a$ is infinitely differentiable in some neighborhood of a and the corresponding Taylor series converges to $f(x)$ in some neighborhood of a (sometimes this is used as the definition of an analytic function). If a function is infinitely differentiable, then it is not necessarily analytic. A well-known example is the Cauchy function $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. At $x = 0$ all derivatives are zero and hence the corresponding Taylor series converges to the zero function.*

Definition D.2 *Let $I \subset \mathbf{R}$ be an open interval. Then f is analytic on I if it is analytic at any point $a \in I$.*

We can extend this definition to closed intervals by making the following changes to the above.

(a) if $f(x)$ is defined for all $x \geq a$, then $f(x)$ is defined to be analytic at *a* if there exist an $r > 0$ and a series of coefficients $\{a_n\}$ such that the equality in (D.1) holds for all $x \geq a$.

(b) if $f(x)$ is defined for all $x > a$, then we say that $f(x)$ can be analytically extended to a if there exist an $r > 0$ and a series of coefficients $\{a_n\}$ such that the equality in (D.1) holds for all $x \geq a$.

The following result is used in the proof of Theorem 3.6 (that the central path has a unique limit point in the optimal set).

Theorem D.1 *Let $f : [0,1) \to \mathbf{R}$ be analytic on* [0,1). *Let $f(0) = 0$ and $x = 0$ be an accumulation point of all x such that $f(x) = 0$. Then $f(x) = 0$ for all $x \in [0,1)$.*

Proof:

Since f is analytic at $x = 0$, there exist $r > 0$ and a_n such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \forall x : 0 \leq x < r. \tag{D.2}$$

We show that $a_n = 0, \forall n$. Let a_k be the first non-zero coefficient in (D.2). Hence

$$f(x) = x^k(a_k + a_{k+1}x + \ldots), \quad \forall x : 0 \leq x < r. \tag{D.3}$$

Let $0 < \rho < r$. The series in (D.3) converges at $x = \rho$ and hence the numbers $a_n\rho^n$ are bounded. Let $|a_n|\rho^n < K$, i.e., $|a_n| < K/\rho^n, \forall n$. Then

$$|f(x)| \geq |x^k|(|a_k| - \frac{K|x|}{\rho^{k+1}} - \frac{K|x|^2}{\rho^{k+2}} - \ldots) = |x^k| \left(|a_k| - \frac{K|x|}{\rho^k(\rho - |x|)} \right), \tag{D.4}$$

for all $0 < x < \rho$. The last expression in (D.4) is positive for all sufficiently small x. This is in contradiction with the assumption that f has roots arbitrarily close to $x = 0$. Hence $a_n = 0, \forall n$ and $f(x) = 0$ in some right neighborhood of 0. Using the analyticity in (0,1) it can be shown that it is zero on the whole domain. □

Appendix E
The (symmetric) Kronecker product

This appendix contains various results about the Kronecker and symmetric Kronecker product. The part on the symmetric Kronecker product is based on Appendix D in Aarts[1].

We will use the **vec** and **svec** notation frequently in this appendix, and restate the definitions here for convenience.

Definition E.1 *For any symmetric $n \times n$ matrix U, the vector $\mathbf{svec}(U) \in \mathbf{R}^{\frac{1}{2}n(n+1)}$ is defined as*

$$\mathbf{svec}(U) = \left(u_{11}, \sqrt{2}u_{21}, \ldots, \sqrt{2}u_{n1}, u_{22}, \sqrt{2}u_{32}, \ldots, \sqrt{2}u_{n2}, \ldots, u_{nn}\right)^T,$$

such that

$$\mathbf{svec}(U)^T\mathbf{svec}(U) = \mathbf{Tr}\left(U^T U\right) = \mathbf{vec}(U)^T\mathbf{vec}(U),$$

where

$$\mathbf{vec}(U) = (u_{11}, u_{21}, \ldots, u_{n1}, u_{12}, \ldots, u_{nn})^T.$$

The inverse map of **svec** is denoted by **smat.**

E.1 THE KRONECKER PRODUCT

Definition E.2 *Let $G \in \mathbf{R}^{m\times n}$ and $K \in \mathbf{R}^{r\times s}$. Then $G \otimes K$ is defined as the $mr \times ns$ matrix with block structure*

$$G \otimes K = [g_{ij}K] \quad i = 1, \ldots, m,\ j = 1, \ldots, n,$$

i.e. *the block in position ij is given by $g_{ij}K$.*

The following identities are proven in Horn and Johnson [86]. (We assume that the sizes of the matrices are such that the relations are defined and that the inverses exist where referenced.)

Theorem E.1 *Let K, L, G and H be real matrices.*

- $G \otimes K\mathbf{vec}(H) = \mathbf{vec}\left(KHG^T\right)$;
- $(G \otimes K)^T = G^T \otimes K^T$;
- $(G \otimes K)^{-1} = G^{-1} \otimes K^{-1}$;
- $(G \otimes K)(H \otimes L) = (GH) \otimes (KL)$;
- *The eigenvalues of* $G \otimes K$ *are given by* $\lambda_i(G)\lambda_j(K)\ \forall\ i,j = 1,\ldots,n$. *As a consequence, if G and K are positive (semi)definite, then so is* $G \otimes K$;
- *If* $Gx_i = \lambda_i(G)x_i$ *and* $Ky_j = \lambda_j(K)y_j$, *then* $\mathbf{vec}\left(y_j x_i^T\right)$ *is an eigenvector of* $G \otimes K$ *with corresponding eigenvalue* $\lambda_i(G)\lambda_j(K)$.

As an application of the Kronecker product we can analyse the solutions of so-called Lyapunov equations.

Theorem E.2 *Let* $A \in \mathbf{R}^{n\times n}$ *and* $B \in \mathcal{S}_n$. *The Lyapunov equation*

$$AX + XA^T = B \tag{E.1}$$

has a unique symmetric solution if A and –A have no eigenvalues in common.

Proof:

Using the first item in Theorem E.1, we can rewrite (E.1) as

$$\mathbf{vec}\left(AX + XA^T\right) = \mathbf{vec}\left(AXI + IXA^T\right) = (I \otimes A + A \otimes I)\,\mathbf{vec}(X) = \mathbf{vec}(B).$$

By using the fourth item of Theorem E. 1 we have

$$(I \otimes A)(A \otimes I) = (IA) \otimes (AI) = (AI) \otimes (IA) = (A \otimes I)(I \otimes A).$$

In other words, the matrices $A \otimes I$ and $I \otimes A$ commute and therefore share a set of eigenvectors. Moreover, by the fifth item of Theorem E. 1 we know that the eigenvalues of $A\otimes I$ and $I\otimes A$ are obtained by taking n copies of the spectrum of *A*. Therefore each eigenvalue of $(I \otimes A + A \otimes I)$ is given by $\lambda_i(A) + \lambda_j(A)$ for some i, j. It follows that the matrix $(I \otimes A + A \otimes I)$ is nonsingular if *A* and –*A* have no eigenvalues in common. In this case equation (E.1) has a unique solution. We must still verify that this solution is symmetric. To this end, note that if *X* is a solution of (E.1), then so is X^T. This completes the proof. □

E.2 THE SYMMETRIC KRONECKER PRODUCT

Definition E.3 *The symmetric Kronecker product of any two $n \times n$ matrices G and K (not necessarily symmetric), is a mapping on a vector $u = \mathbf{svec}(U)$ where U is a symmetric $n \times n$ matrix and is defined as*

$$(G \otimes_s K)u = \frac{1}{2}\mathbf{svec}(KUG^T + GUK^T). \tag{E.2}$$

Note that the linear operator $G \otimes_s K$ is defined implicitly in (E.2). We can give a matrix representation of $G \otimes_s K$ by introducing the orthogonal $\frac{1}{2}n(n+1) \times n$ matrix Q(*i.e.* $QQ^T = I_{\frac{1}{2}n(n+1)}$), with the property that

$$Q\mathbf{vec}(U) = \mathbf{svec}(U) \text{ and } Q^T\mathbf{svec}(U) = \mathbf{vec}(U) \quad \forall U \in \mathcal{S}_n. \tag{E.3}$$

Theorem E.3 *Let Q be the unique orthogonal $\frac{1}{2}n(n+1) \times n$ matrix that satisfies (E.3). For any $G \in \mathbf{R}^{n\times n}$ and $K \in \mathbf{R}^{n\times n}$ one has*

$$G \otimes_s K = \frac{1}{2}Q\left(G \otimes K + K \otimes G\right)Q^T.$$

Proof:

Let $U \in \mathcal{S}_n$ be given and note that

$$\begin{aligned}
\frac{1}{2}Q\left(G \otimes K + K \otimes G\right)Q^T\mathbf{svec}(U) &= \frac{1}{2}Q\left(G \otimes K + K \otimes G\right)\mathbf{vec}(U) \\
&= \frac{1}{2}Q\left((G \otimes K)\mathbf{vec}(U) + (K \otimes G)\mathbf{vec}(U)\right) \\
&= \frac{1}{2}Q\left(\mathbf{vec}\left(KUG^T\right) + \mathbf{vec}\left(GUK^T\right)\right) \\
&= \frac{1}{2}Q\left(\mathbf{vec}\left(KUG^T + \left(KUG^T\right)^T\right)\right) \\
&= \frac{1}{2}\mathbf{svec}\left(KUG^T + GUK^T\right) \\
&= (G \otimes_s K)\mathbf{svec}(U),
\end{aligned}$$

where we have used the first identity in Theorem E. 1 to obtain the third equality. □

Definition E.4 *If for every vector $u = \mathbf{svec}(U)$ where U is a symmetric nonzero matrix,*

$$u^T(G \otimes_s K)u > 0,$$

then $(G \otimes_s K)$ is called positive definite.

Lemma E.1 *The symmetric Kronecker product has the following properties.*

1. $(G \otimes_s K) = (K \otimes_s G)$;
2. $(G \otimes_s K)(H \otimes_s L) = \frac{1}{2}(GH \otimes_s KL + GL \otimes_s KH)$;
3. $\mathbf{svec}(U)^T \mathbf{svec}(V) = \mathbf{Tr}\,(UV) = \mathbf{Tr}\,(VU)$ *for two symmetric matrices U and V;*
4. *If G and K are symmetric positive definite, then $(G \otimes_s K)$ is positive definite;*
5. $(G \otimes_s K)^T = (G^T \otimes_s K^T)$.

Proof:

1. This directly follows from Definition E.3.
2. Let U be a symmetric matrix, then

$$
\begin{aligned}
& (G \otimes_s K)(H \otimes_s L)\mathbf{svec}(U) \\
= \;& \frac{1}{2}(G \otimes_s K)\mathbf{svec}(HUL^T + LUH^T) \\
= \;& \frac{1}{4}\mathbf{svec}\big(GHUL^TK^T + KHUL^TG^T + GLUH^TK^T + KLUH^TG^T\big) \\
= \;& \frac{1}{4}\mathbf{svec}\big(GHU(KL)^T + KLU(GH)^T + KHU(GL)^T + GLU(KH)^T\big) \\
= \;& \frac{1}{2}(GH \otimes_s KL + GL \otimes_s KH)\mathbf{svec}(U).
\end{aligned}
$$

3. See Definition E.1.
4. For every symmetric nonzero matrix U we have to prove that

$$u^T(G \otimes_s K)u > 0,$$

where $u = \mathbf{svec}(U)$. Now

$$u^T(G\otimes_s K)u = \frac{1}{2}u^T\mathbf{svec}(GUK + KUG) = \frac{1}{2}\mathbf{svec}(U)^T\mathbf{svec}(GUK + KUG),$$

since G and K are symmetric. By using Property 3 we derive

$$\frac{1}{2}(\mathbf{svec}(U))^T\mathbf{svec}(GUK + KUG) = \frac{1}{2}(\mathbf{Tr}\,(UGUK) + \mathbf{Tr}\,(UKUG)).$$

Since U is nonzero and K and G are symmetric positive definite and therefore nonsingular, we obtain that $K^{\frac{1}{2}}UG^{\frac{1}{2}} \neq 0$ and thus

$$\frac{1}{2}(\mathbf{Tr}\,(UGUK) + \mathbf{Tr}\,(UKUG)) = \mathbf{Tr}\,(UKUG) = \left\|K^{\frac{1}{2}}UG^{\frac{1}{2}}\right\|^2 > 0.$$

5. Define $u = \mathbf{svec}(U)$ and $\ell = \mathbf{svec}(L)$ for arbitrary symmetric matrices U and L and $m = (G \otimes_s K)u$. Now

$$\ell^T m = (\mathbf{svec}(L))^T (G \otimes_s K)u = \frac{1}{2}(\mathbf{svec}(L))^T \mathbf{svec}(GUK^T + KUG^T)$$

and by using Property 3 it follows that

$$\begin{aligned}
& \frac{1}{2}(\mathbf{svec}(L))^T \mathbf{svec}(GUK^T + KUG^T) \\
= \; & \frac{1}{2}\mathbf{Tr}\,(LGUK^T + LKUG^T) \\
= \; & \frac{1}{2}\mathbf{Tr}\,(UK^T LG + UG^T LK) \\
= \; & \frac{1}{2}(\mathbf{svec}(U))^T \mathbf{svec}(K^T LG + G^T LK) \\
= \; & (\mathbf{svec}(U))^T (G^T \otimes_s K^T)\mathbf{svec}(L) \\
= \; & (\mathbf{svec}(U))^T (G^T \otimes_s K^T)\ell.
\end{aligned}$$

Since $\ell^T m = m^T \ell$, we obtain

$$m^T = (\mathbf{svec}(U))^T (G^T \otimes_s K^T)$$

and from the definition of m we derive

$$m^T = ((G \otimes_s K)\mathbf{svec}(U))^T = \mathbf{svec}(U)^T (G \otimes_s K)^T$$

and thus

$$(G \otimes_s K)^T = (G^T \otimes_s K^T).$$

□

Definition E.5 *For every nonsingular matrix P we define the operators $\mathcal{E}$ and $\mathcal{F}$ as*

$$\mathcal{E} = (P \otimes_s P^{-T} S), \qquad \mathcal{F} = (PX \otimes_s P^{-T}).$$

By using Property 2 in Lemma E. 1 we obtain

$$\mathcal{E} = (P \otimes_s P^{-T} S) = (I \otimes_s P^{-T} S P^{-1})(P \otimes_s P)$$

and

$$\mathcal{F} = (PX \otimes_s P^{-T}) = (PXP^T \otimes_s I)(P^{-T} \otimes_s P^{-T}).$$

We define the operator H_P for any nonsingular $n \times n$ matrix P as

$$H_P(Q) = \frac{1}{2}(PQP^{-1} + P^{-T} Q^T P^T).$$

The next theorems are based on Theorem 3.1 and Theorem 3.2 in Todd *et al.* [173].

Theorem E.4 *If the matrices X and S are positive definite and the matrix $H_P(XS)$ is symmetric positive semidefinite, then $\mathcal{E}^{-1}\mathcal{F}$ is positive definite.*

Proof:

Consider an arbitrary nonzero vector $g \in \mathbf{R}^{\frac{n(n+1)}{2}}$. We now prove that $g^T\mathcal{E}^{-1}\mathcal{F}g > 0$. Defining $k := \mathcal{E}^{-T}g$ and $K = \mathbf{smat}(k)$, where k is also nonzero since $\mathcal{E}^{-T}$ exists, it follows by using Property 5 in Lemma E. 1 that

$$g^T\mathcal{E}^{-1}\mathcal{F}g = k^T\mathcal{F}\mathcal{E}^Tk = k^T(PX \otimes_s P^{-T})(P^T \otimes_s SP^{-1})k.$$

By using Property 2 in Lemma E.1 we derive

$$\begin{aligned} k^T(PX \otimes_s P^{-T})(P^T \otimes_s SP^{-1})k &= \frac{1}{2}k^T(PXP^T \otimes_s P^{-T}SP^{-1})k \\ &\quad + \frac{1}{2}k^T(PXSP^{-1} \otimes_s I)k. \end{aligned}$$

Since *X* and *S* are symmetric positive definite, PXP^T and $P^{-T}SP^{-1}$ are symmetric positive definite and by using Property 4 in Lemma E. 1 it follows that $(PXP^T \otimes_s P^{-T}SP^{-1})$ is positive definite. Therefore

$$\begin{aligned} & \frac{1}{2}k^T(PXP^T \otimes_s P^{-T}SP^{-1})k + \frac{1}{2}k^T(PXSP^{-1} \otimes_s I)k \\ > \; & \frac{1}{2}k^T(PXSP^{-1} \otimes_s I)k \\ = \; & \frac{1}{4}(\mathbf{svec}(K))^T\mathbf{svec}(PXSP^{-1}K + KP^{-T}SXP^T). \end{aligned}$$

By using Property 3 in Lemma E. 1 we now obtain

$$\begin{aligned} & \frac{1}{4}\mathbf{svec}(K)^T\mathbf{svec}\left(PXSP^{-1}K + KP^{-T}SXP^T\right) \\ = \; & \frac{1}{4}\mathbf{Tr}\,\left(K\left(PXSP^{-1} + P^{-T}SXP^T\right)K\right) \\ = \; & \frac{1}{2}\mathbf{Tr}\,(KH_P(XS)K). \end{aligned}$$

Since we assumed that $H_P(XS)$ is positive semidefinite and the matrix *K* is symmetric, it follows that the matrix $KH_P(XS)K$ is positive semidefinite. Thus

$$\frac{1}{2}\mathbf{Tr}\,(KH_P(XS)K) \geq 0.$$

□

Theorem E.5 *If the matrices X and S are symmetric positive definite and the matrices PXP^T and $P^{-T}SP^{-1}$ commute, then $H_P(XS)$ is symmetric positive definite.*

Proof:

If PXP^T and $P^{-T}SP^{-1}$ commute, it follows that

$$PXSP^{-1} = (PXP^T)(P^{-T}SP^{-1}) = (P^{-T}SP^{-1})(PXP^T) = P^{-T}SXP^T.$$

Therefore the matrix $PXSP^{-1}$ is symmetric and

$$H_P(XS) = \frac{1}{2}(PXSP^{-1} + P^{-T}SXP^T) = PXSP^{-1}.$$

From Theorem A.4 we know that XS — and therefore $PXSP^{-1}$ — have positive eigenvalues. Since $H_P(XS)$ is symmetric it is therefore positive definite. □

Appendix F
Search directions for the embedding problem

Conditions for the existence and uniqueness of several search directions for the self–dual embedding problem of Chapter 4 (Section 4.2) are derived here.

The feasible directions of interior point methods for the embedding problem (4.4) can be computed from the following generic linear system:

$$
\begin{array}{lllllllll}
 & \mathbf{Tr}\,(A_i\Delta X) & -\Delta\tau b_i & +\Delta\theta\bar{b}_i & & & & = 0 \\
-\sum_{i=1}^m \Delta y_i A_i & & +\Delta\tau C & -\Delta\theta\bar{C} & -\Delta S & & & = 0 \\
b^T\Delta y & -\mathbf{Tr}\,(C\Delta X) & & +\Delta\theta\alpha & & -\Delta\rho & & = 0 \\
-\bar{b}^T\Delta y & +\mathbf{Tr}\,(\bar{C}\Delta X) & -\Delta\tau\alpha & & & & -\Delta\nu & = 0
\end{array}
\tag{F.1}
$$

where $i = 1, \ldots, m,$ and

$$
\left.
\begin{array}{rcl}
H_P(\Delta XS + X\Delta S) & = & \mu I - H_P(XS), \\
\rho\Delta\tau + \tau\Delta\rho & = & \mu - \tau\rho \\
\nu\Delta\theta + \theta\Delta\nu & = & \mu - \theta\nu,
\end{array}
\right\}
\tag{F.2}
$$

where H_P is the linear transformation given by

$$
H_P(M) := \frac{1}{2}\left[PMP^{-1} + P^{-T}M^TP^T\right],
$$

for any matrix *M,* and where the *scaling matrix P* determines the symmetrization strategy. The best-known choices of *P* from the literature are listed in Table F. 1. We will now prove (or derive sufficient conditions) for existence and uniqueness of the search directions corresponding to each of the choices of *P* in Table F. 1. To this end,

P	Reference
$\left[X^{\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}}$	Nesterov and Todd [138];
$X^{-\frac{1}{2}}$	Monteiro [124], Kojima *et al.* [108];
$S^{\frac{1}{2}}$	Monteiro [124], Helmberg *et al.* [84], Kojima *et al.* [108];
I	Alizadeh, Haeberley and Overton [6];

Table F.1. Choices for the scaling matrix P.

we will write the equations (F.1) and (F.2) as a single linear system and show that the coefficient matrix of this system is nonsingular.[1]

We will use the notation

$$\tilde{A}_i := \begin{bmatrix} A_i & & \\ & -b_i & \\ & & \bar{b}_i \end{bmatrix} \quad (i = 1, \ldots, m),$$

and define $\tilde{P}$ by replacing X by $\tilde{X}$ and S by $\tilde{S}$ in Table F.1. We will rewrite (F.2) by using the *symmetric Kronecker product.* For a detailed review of the Kronecker product see Appendix E; we only restate the relevant definitions here for convenience.

- $\mathbf{svec}(X) := [X_{11},\ \sqrt{2}X_{12},\ \ldots\ \sqrt{2}X_{1n},\ X_{22},\ \sqrt{2}X_{23},\ \ldots,\ X_{nn}]^T$;
- The *symmetric Kronecker product* $G \otimes_s K$ of $G, K \in \mathbb{R}^{n\times n}$ is implicitly defined via

$$(G \otimes_s K)\,\mathbf{svec}(H) := \frac{1}{2}\mathbf{svec}\left(KHG^T + GHK^T\right) \ (\forall\ H \in \mathcal{S}^n).$$

Using the symmetric Kronecker notation, we can combine (F.1) and (F.2) as

$$\begin{bmatrix} 0 & \tilde{\mathcal{A}} & 0 \\ -\tilde{\mathcal{A}}^T & S_{kew} & I \\ 0 & E & F \end{bmatrix} \begin{bmatrix} \Delta y \\ \mathbf{svec}\left(\Delta\tilde{X}\right) \\ \mathbf{svec}\left(\Delta\tilde{S}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{svec}\left(\mu I - H_{\tilde{P}}(\tilde{X}\tilde{S})\right) \end{bmatrix} \tag{F.3}$$

[1] The approach used here is a straightforward extension of the analysis by Todd *et al.* in [173], where this result was proved for SDP problems in the standard form (P) and (*D*).

where

$$\begin{aligned}
\tilde{\mathcal{A}} &:= [\mathbf{svec}(\tilde{A}_1)\ \ldots\ \mathbf{svec}(\tilde{A}_m)]^T \\
\mathcal{S}_{kew} &:= \begin{bmatrix} 0 & \mathbf{svec}(C) & -\mathbf{svec}(\bar{C}) \\ -\mathbf{svec}(C)^T & 0 & \alpha \\ \mathbf{svec}(\bar{C})^T & -\alpha & 0 \end{bmatrix} \\
E &:= \tilde{P} \otimes_s (\tilde{P}^{-T}\tilde{S}), \quad F := (\tilde{P}\tilde{X}) \otimes_s \tilde{P}^{-T}.
\end{aligned}$$

By Theorem E.4 in Appendix E we have the following result.

Lemma F.1 (Toh *et al.* [175]) *Let $\tilde{P}$ be invertible and $\tilde{X}$ and $\tilde{S}$ symmetric positive definite. Then the matrices E and F are invertible. If one also has $H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0$, then the symmetric part of $E^{-1}F$ is also positive definite.*

We are now in a position to prove a sufficient condition for uniqueness of the search direction.

Theorem F.1 *The linear system (F.3) has a unique solution if $H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0$.*

Proof:

We consider the homogeneous system

$$\begin{bmatrix} 0 & \tilde{\mathcal{A}} & 0 \\ -\tilde{\mathcal{A}}^T & \mathcal{S}_{kew} & I \\ 0 & E & F \end{bmatrix} \begin{bmatrix} \Delta y \\ \mathbf{svec}\left(\Delta\tilde{X}\right) \\ \mathbf{svec}\left(\Delta\tilde{S}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{F.4}$$

and prove that it has only the zero vector as solution.

From (F.4) we have

$$\mathbf{svec}\left(\Delta\tilde{S}\right) = \tilde{\mathcal{A}}^T\Delta y - \mathcal{S}_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right)$$

and

$$\mathbf{svec}\left(\Delta\tilde{S}\right) = -\left(F^{-1}E\right)\mathbf{svec}\left(\Delta\tilde{X}\right). \tag{F.5}$$

Eliminating $\mathbf{svec}\left(\Delta\tilde{S}\right)$ from the last two equations gives

$$\tilde{\mathcal{A}}^T\Delta y - \mathcal{S}_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right) + \left(F^{-1}E\right)\mathbf{svec}\left(\Delta\tilde{X}\right) = 0. \tag{F.6}$$

System (F.4) also implies

$$\tilde{\mathcal{A}}\mathbf{svec}\left(\Delta\tilde{X}\right) = 0. \tag{F.7}$$

From (F.6) we have

$$\begin{aligned}\mathbf{svec}\left(\Delta\tilde{X}\right)^T \tilde{A}^T\Delta y - \mathbf{svec}\left(\Delta\tilde{X}\right)^T S_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right) & \\ +\mathbf{svec}\left(\Delta\tilde{X}\right)^T \left(F^{-1}E\right)\mathbf{svec}\left(\Delta\tilde{X}\right) &= 0.\end{aligned}$$

The first term on the left-hand side is zero, by (F.7), and the second term is zero by the skew-symmetry of S_{kew}. We therefore have

$$\mathbf{svec}\left(\Delta\tilde{X}\right)^T \left(F^{-1}E\right)\mathbf{svec}\left(\Delta\tilde{X}\right) = 0,$$

which shows that $\Delta\tilde{X} = 0$, since EF^{-1} is assumed to be (non-symmetric) positive definite. It follows that $\Delta\tilde{S} = 0$ by (F.5). Furthermore, $\Delta y = 0$ by (F.6), since $\tilde{A}$ has full rank (the matrices A_i $(i = 1, \ldots, m)$ are linearly independent). □

All that remains is to analyze the condition

$$H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0 \tag{F.8}$$

in the theorem. For the first three choices of $\tilde{P}$ in Table F.1, condition (F.8) always holds (by Theorem E.5 in Appendix E). For $\tilde{P} = I$ (the so-called AHO direction), (F.8) becomes the condition $\tilde{X}\tilde{S} + \tilde{S}\tilde{X} \succ 0$.

An alternative sufficient condition for existence of the AHO direction was derived by Monteiro and Zanjácomo [126], namely

$$\left\|\frac{1}{\mu}\tilde{X}^{\frac{1}{2}}\tilde{S}\tilde{X}^{\frac{1}{2}} - I\right\| \leq \frac{1}{2},$$

where $\mu = \mathbf{Tr}\left(\tilde{X}\tilde{S}\right)/\tilde{n}$.

Appendix G
Regularized duals

In this appendix we review some strong duality results for SDP problems that do not satisfy the Slater condition.

The duals in question are obtained through a procedure called regularization. Although a detailed treatment of regularization is beyond the scope of this monograph, the underlying idea is quite simple:[1] if the problem

$$(D): \quad d^* := \sup_{y,S} \left\{ b^T y \;\middle|\; \sum_{i=1}^{m} y_i A_i + S = C,\ S \succeq 0,\ y \in \mathbf{R}^m \right\}$$

is feasible, but not strictly feasible, we can obtain a 'strictly feasible reformulation' by replacing the semidefinite cone by a suitable lower dimensional face (say $\mathcal{F} \subset \mathcal{S}_n^+$) of it, such that the new problem

$$(\bar{D}): \quad d^* := \sup_{y,S} \left\{ b^T y \;\middle|\; \sum_{i=1}^{m} y_i A_i + S = C,\ S \in \mathcal{F},\ y \in \mathbf{R}^m \right\}$$

is strictly feasible in the sense that there exists a pair (y, S) such that $S \in \mathrm{ri}\,(\mathcal{F})$ and $\sum_{i=1}^{m} y_i A_i + S = C$. Problem $(\bar{D})$ will have a perfect dual, by the conic duality theorem (Theorem 2.3). The main point is to find an explicit expression of the dual of the face $\mathcal{F}$. The resulting dual problem now takes the form:

$$(P') \quad \inf_X \left\{ \mathbf{Tr}\,(CX) \mid \mathbf{Tr}\,(A_i X) = b_i \ (i = 1, \ldots, m),\ X \in \mathcal{F}^* \right\},$$

where $\mathcal{F}^*$ denotes the dual cone of $\mathcal{F}$. In the SDP case $\mathcal{F}^*$ can be described by a system of linear matrix inequalities. There is more than one way to do this and one can obtain different dual problems within this framework (see Pataki [144] for details).

[1] The idea of regularization was introduced by Borwein and Wolkowicz [30]. For a more recent (and simplified) treatment the reader is referred to the excellent exposition by Pataki [144].

Ramana [153] first obtained a regularized dual for (*D*);[2] the so-called gap-free (or extended Lagrange–Slater) primal dual (P_{gf}) of (*D*) takes the form:

$$p_{gf}^* := \inf \mathbf{Tr}\ (C(U_0 + W_L))$$

subject to

$$\begin{array}{rcl} \mathbf{Tr}\ (A_k(U_0 + W_L)) & = & b_k, \quad k = 1, \ldots, m \\ \mathbf{Tr}\ (C(U_i + W_{i-1})) & = & 0, \quad i = 1, \ldots, L \\ \mathbf{Tr}\ (A_k(U_i + W_{i-1})) & = & 0, \quad i = 1, \ldots, L,\ k = 1, \ldots, m \\ W_0 & = & 0 \\ \begin{bmatrix} I & W_i^T \\ W_i & U_i \end{bmatrix} & \succeq & 0, \quad i = 1, \ldots, L \\ U_0 & \succeq & 0, \end{array}$$

where the variables are $U_i \succeq 0$ and $W_i \in \mathbf{R}^{n \times n}$, $i = 0, \ldots, L$, and

$$L = \min \left\{ n,\ \frac{1}{2}n(n+1) - m - 1 \right\}.$$

Note that the gap-free primal problem is easily cast in the standard primal form. Moreover, its size is polynomial in the size of (*D*). Unlike the Lagrangian dual (*P*) of (*D*), (P_{gf}) has the following desirable features:

- (Weak duality) If $(y, S) \in \mathcal{D}$ and (U_i, W_i) $(i = 0, \ldots, m)$ is feasible for (P_{gf}), then
$$b^T y \le \mathbf{Tr}\ (C(U_0 + W_m)).$$
- (Dual boundedness) If (*D*) is feasible, its optimal value is finite if and only if (P_{gf}) is feasible.
- (Zero duality gap) The optimal value p_{gf}^* of (P_{gf}) equals the optimal value of (*D*) if and only if both (P_{gf}) and (*D*) are feasible.
- (Attainment) If the optimal value of (*D*) is finite, then it is attained by (P_{gf}).

The standard (Lagrangian) dual problem associated with (P_{gf}) is called the *corrected dual* (D_{cor}). The pair (P_{gf}) and (D_{cor}) are now in perfect duality; see Ramana and Freund[154].

Moreover, a feasible solution to (*D*) can be extracted from a feasible solution to (D_{cor}). The only problem is that (D_{cor}) does not necessarily attain its supremum, even if (*D*) does.

[2] In fact, Ramana did not derive this dual via regularization initially; it was only shown subsequently that it can be derived in this way in Ramana *et al.* [155].

Example G.1 *It is readily verified that the weakly infeasible problem* (D) *in Example 2.2 has a weakly infeasible corrected problem* (D_{cor}).

The possible duality relations are listed in Table G.1. The optimal value of D_{cor} is denoted by d^*_{cor}.

Status of (D)	Status of (P_{gf})	Status of (D_{cor})
$d^* < \infty$	$p^*_{gf} = d^*$	$d^*_{cor} = d^*$
unbounded	infeasible	unbounded
infeasible	unbounded	infeasible

Table G.1. Duality relations for a given problem (D), its gap-free dual (P_{gf}) and its corrected problem (D_{cor}).

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