Supplement to "Extended Comparisons of Best Subset Selection, Forward Stepwise Selection, and the Lasso"

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This supplementary document contains plots from the simulation suite described in the paper "Extended Comparisons of Best Subset Selection, Forward Stepwise Selection, and the Lasso". The plots in Section A precisely follow the simulation format described in the paper. Those in Section B follow an analogous format, except that the tuning has been done using an "oracle", rather than a validation set as in Section A. Specifically, the tuning parameter for each method in each scenario is chosen to minimize the average risk over all of the repetitions. Sections A and B compare best subset selection, forward stepwise regression, the lasso and the relaxed lasso. In Sections C and D we have added three more methods to the comparisons: L0Learn 1 (pure ℓ_0 penalty), L0Learn 2 (mixture of ℓ_0 and ℓ_1 penalties), and SparseNet.

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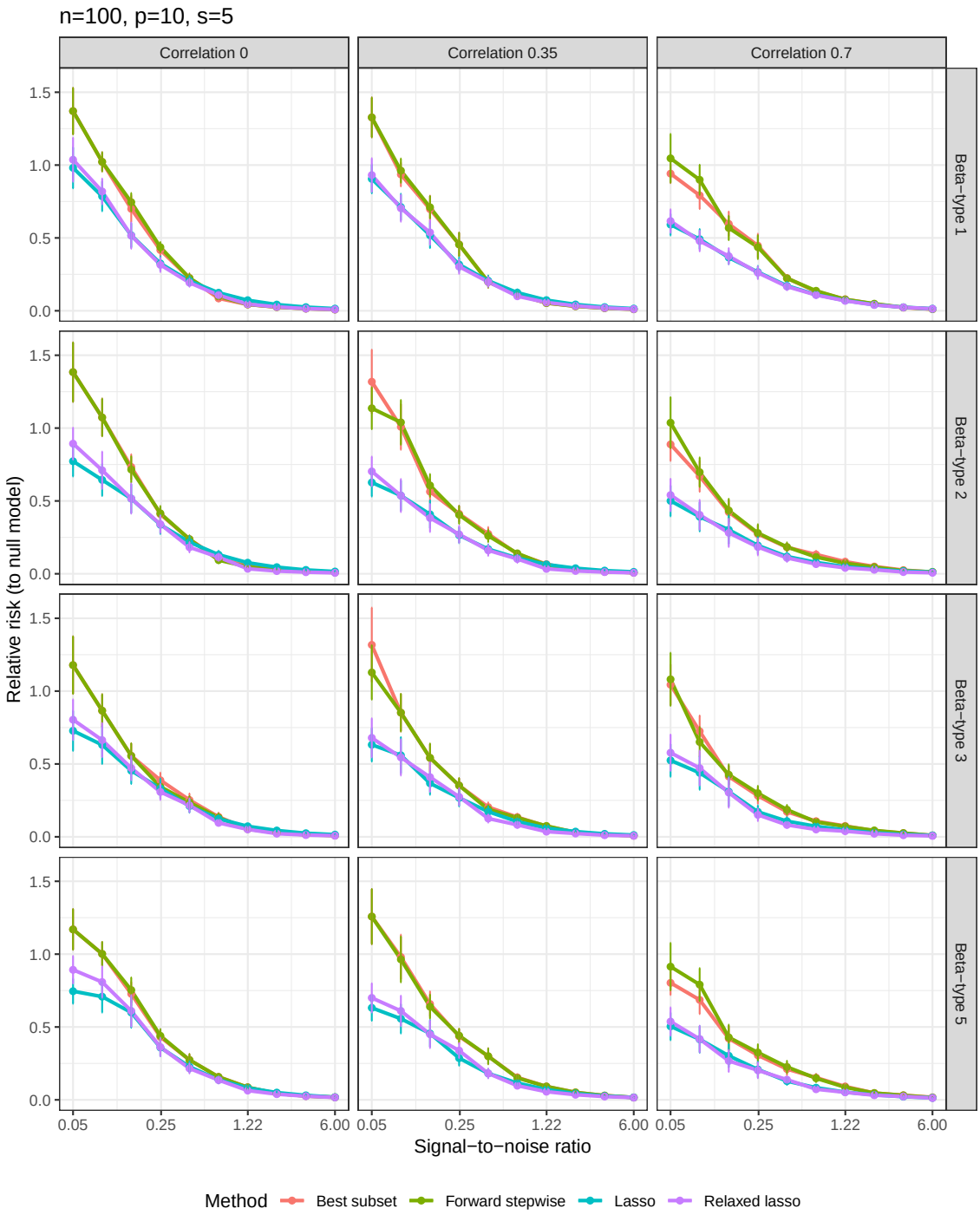
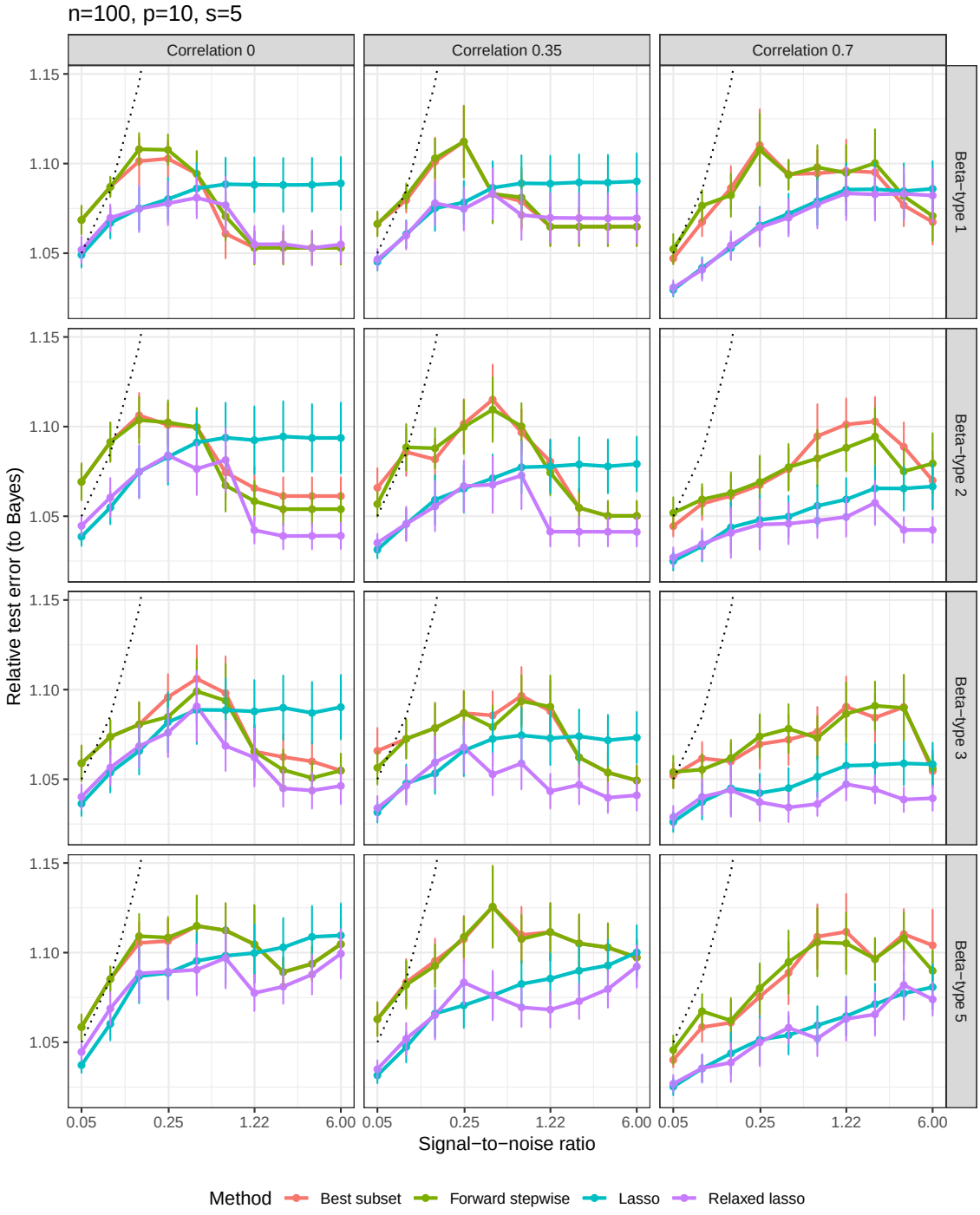
A Validation tuning

A.1 Low setting: $n = 100$, $p = 10$, $s = 5$

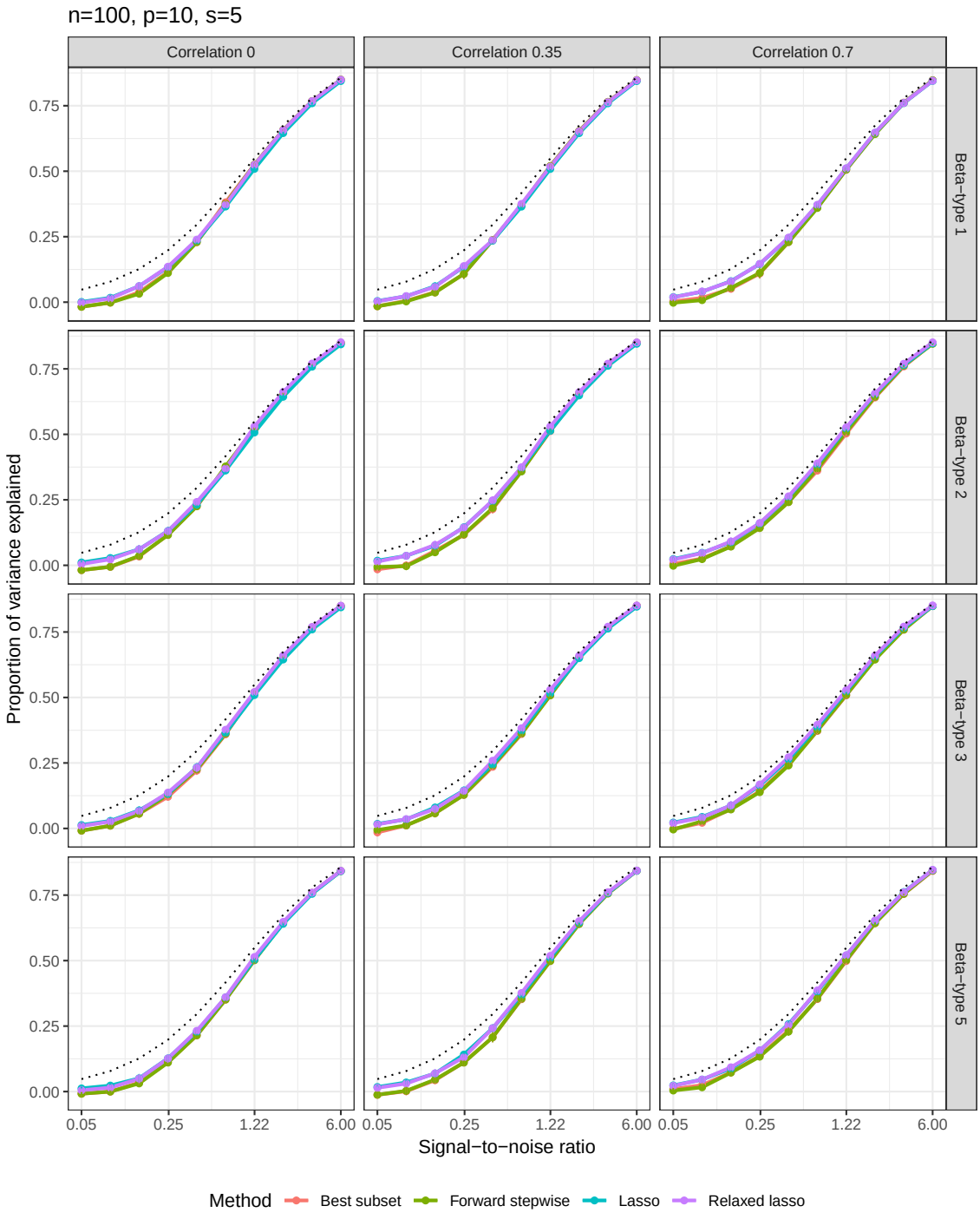
A.1.1 Relative risk (to null model)



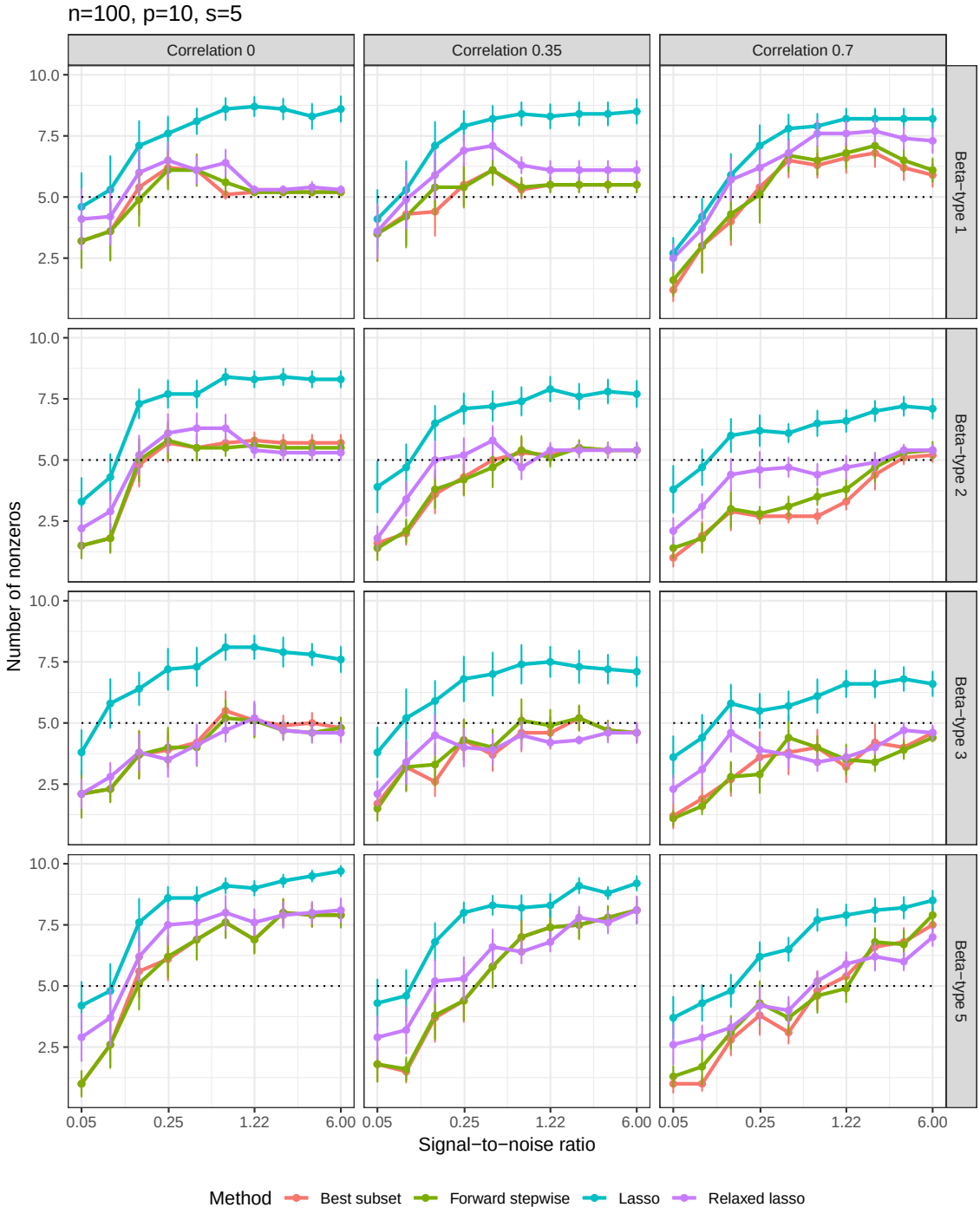
A.1.2 Relative test error (to Bayes)



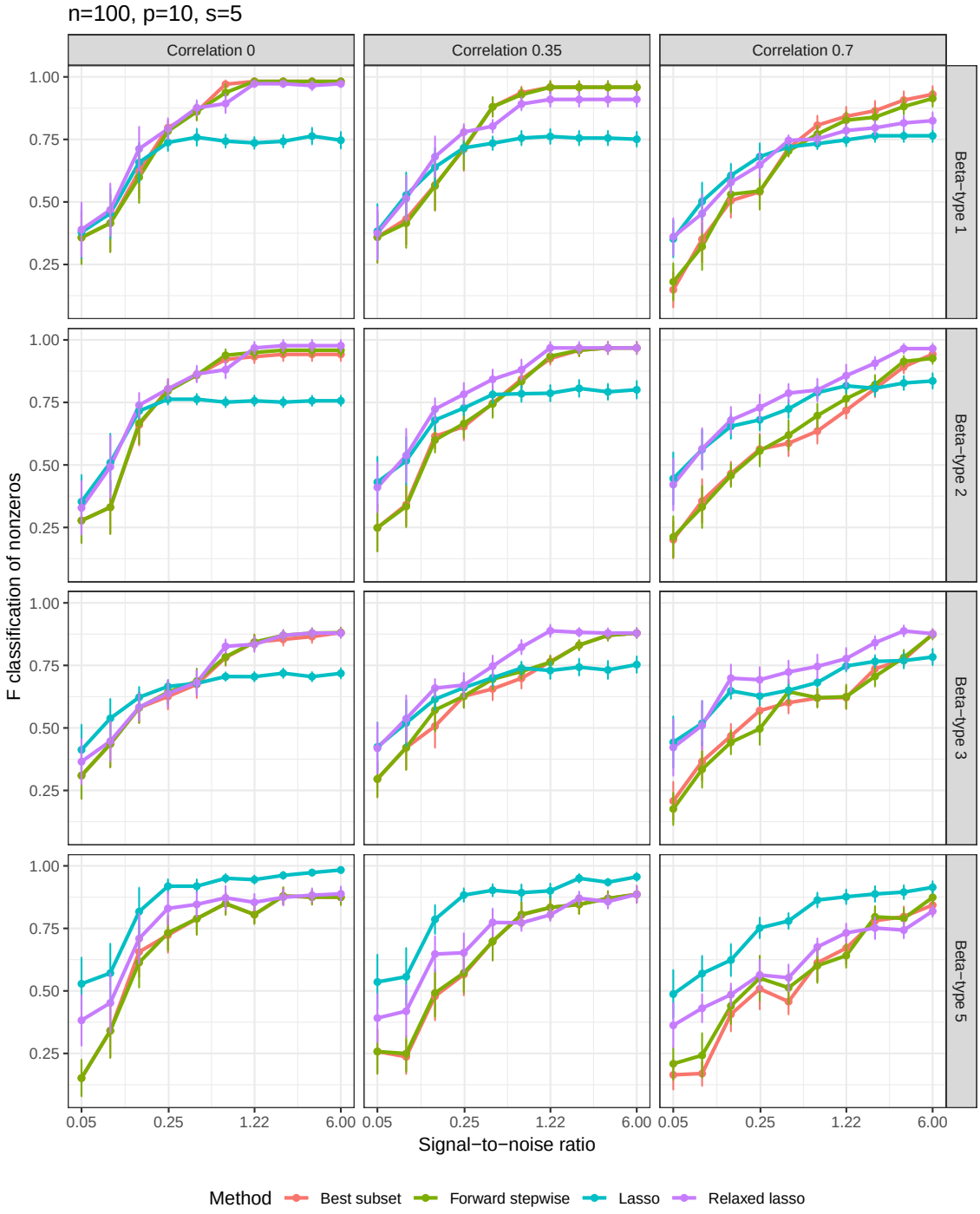
A.1.3 Proportion of variance explained



A.1.4 Number of nonzero coefficients

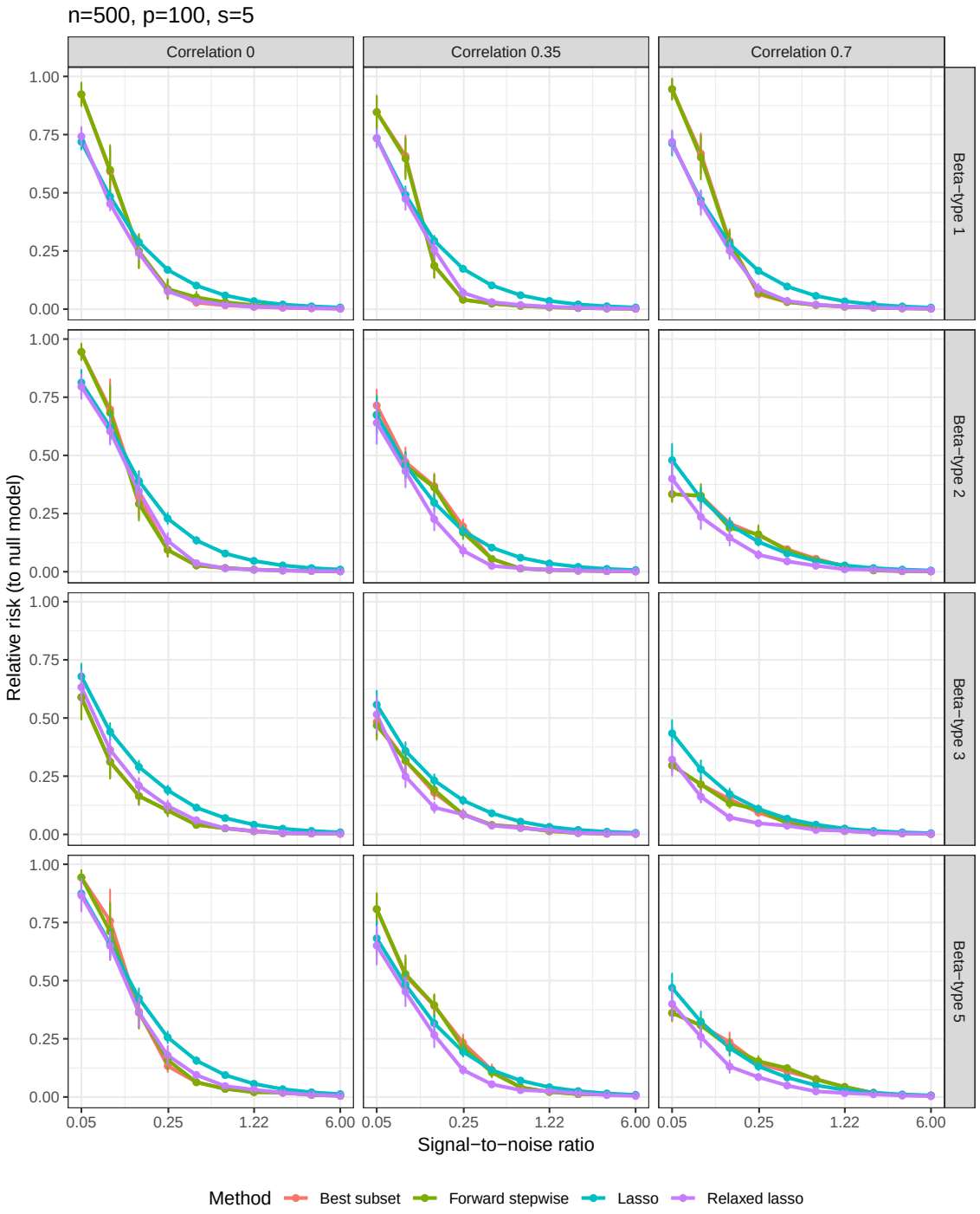


A.1.5 F-score

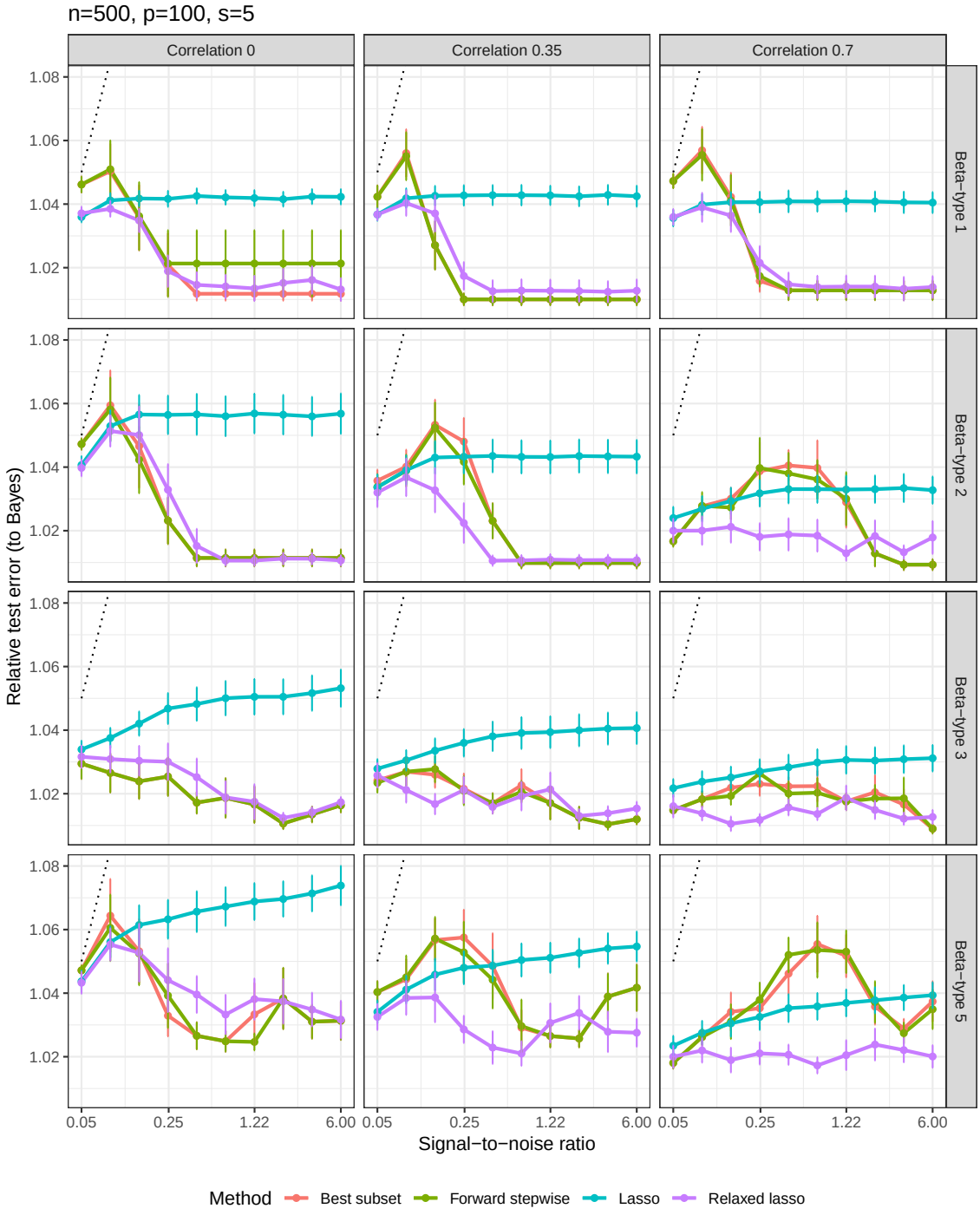

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

A.2 Medium setting: $n = 500$, $p = 100$, $s = 5$

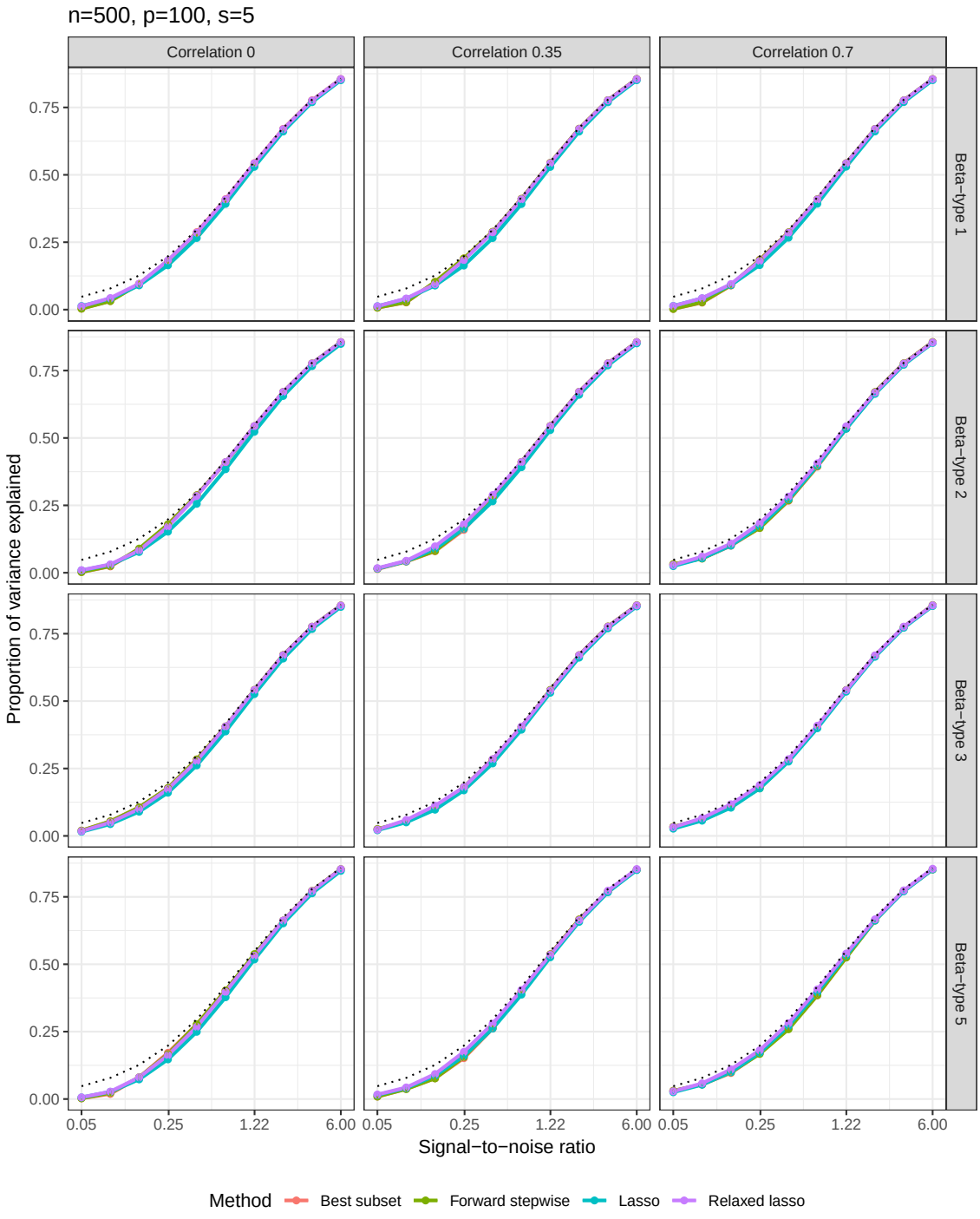
A.2.1 Relative risk (to null model)



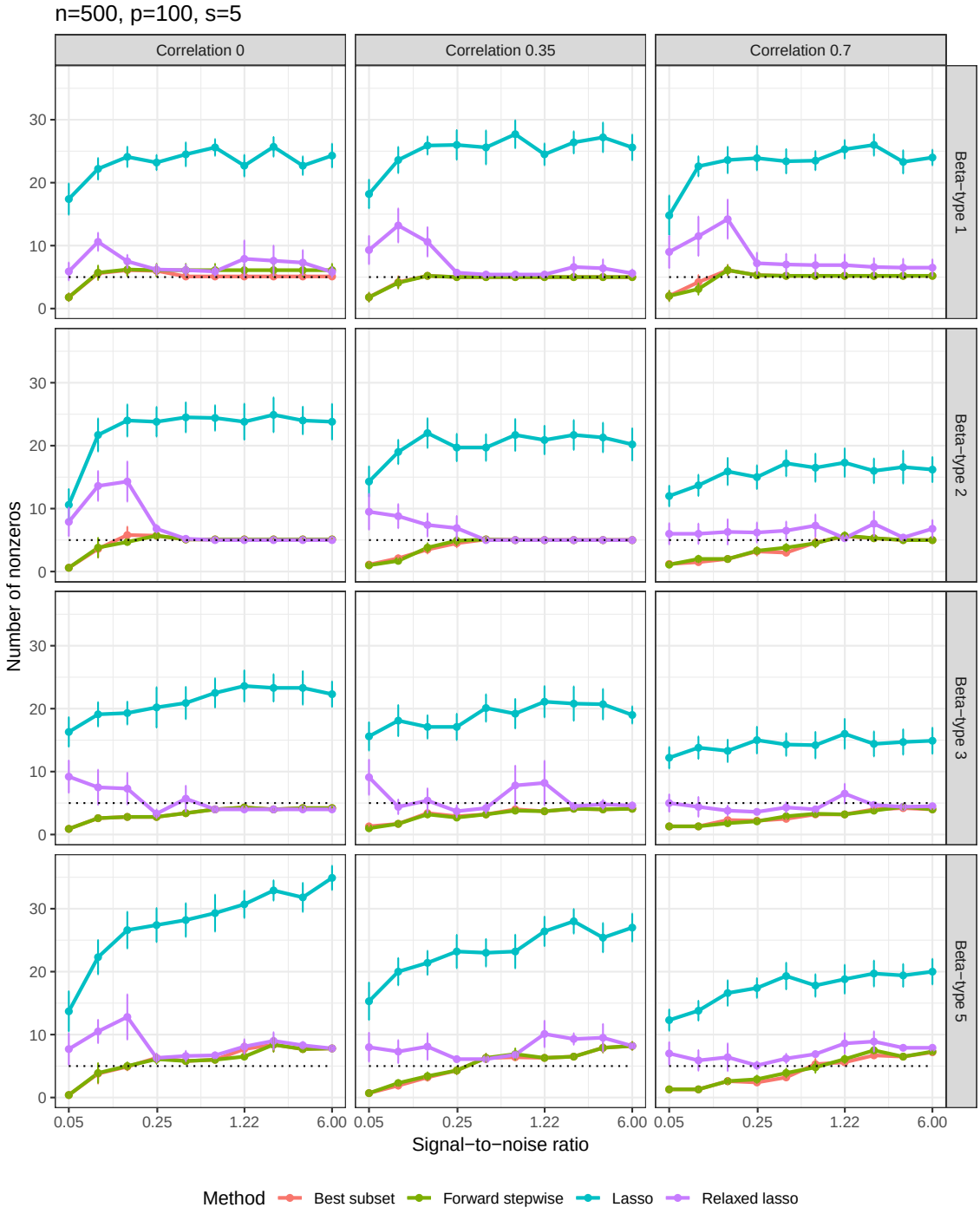
A.2.2 Relative test error (to Bayes)



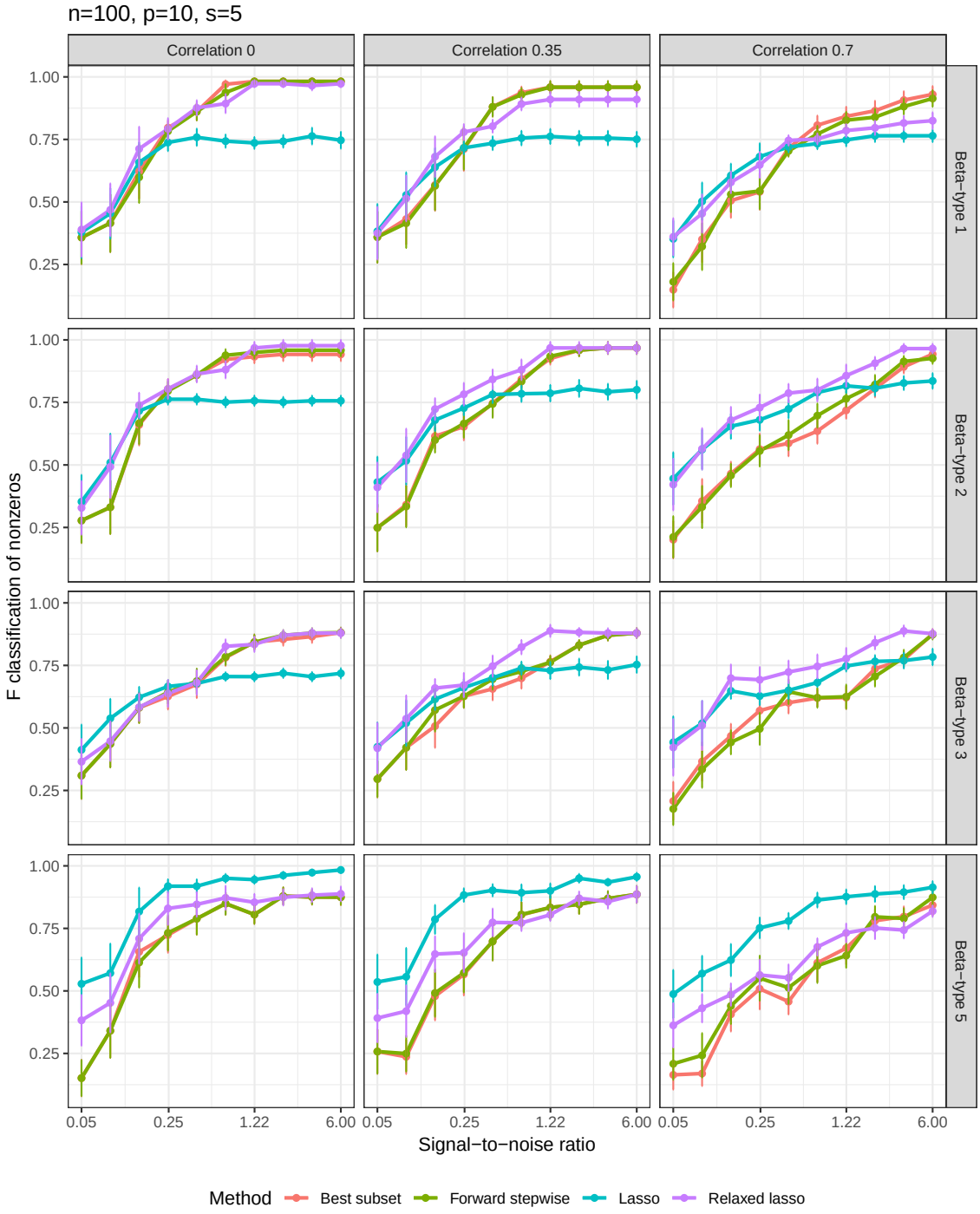
A.2.3 Proportion of variance explained



A.2.4 Number of nonzero coefficients

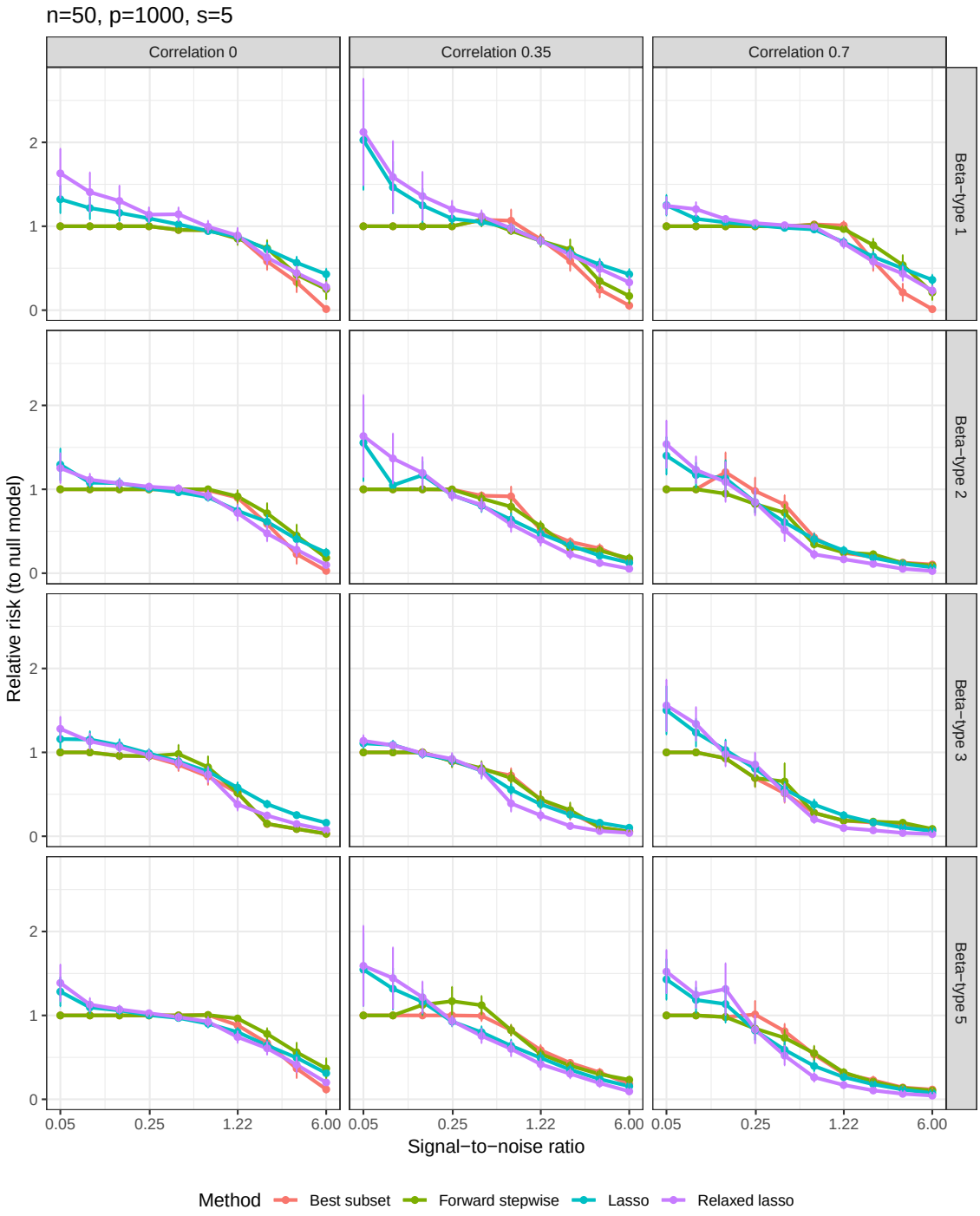


A.2.5 F-score

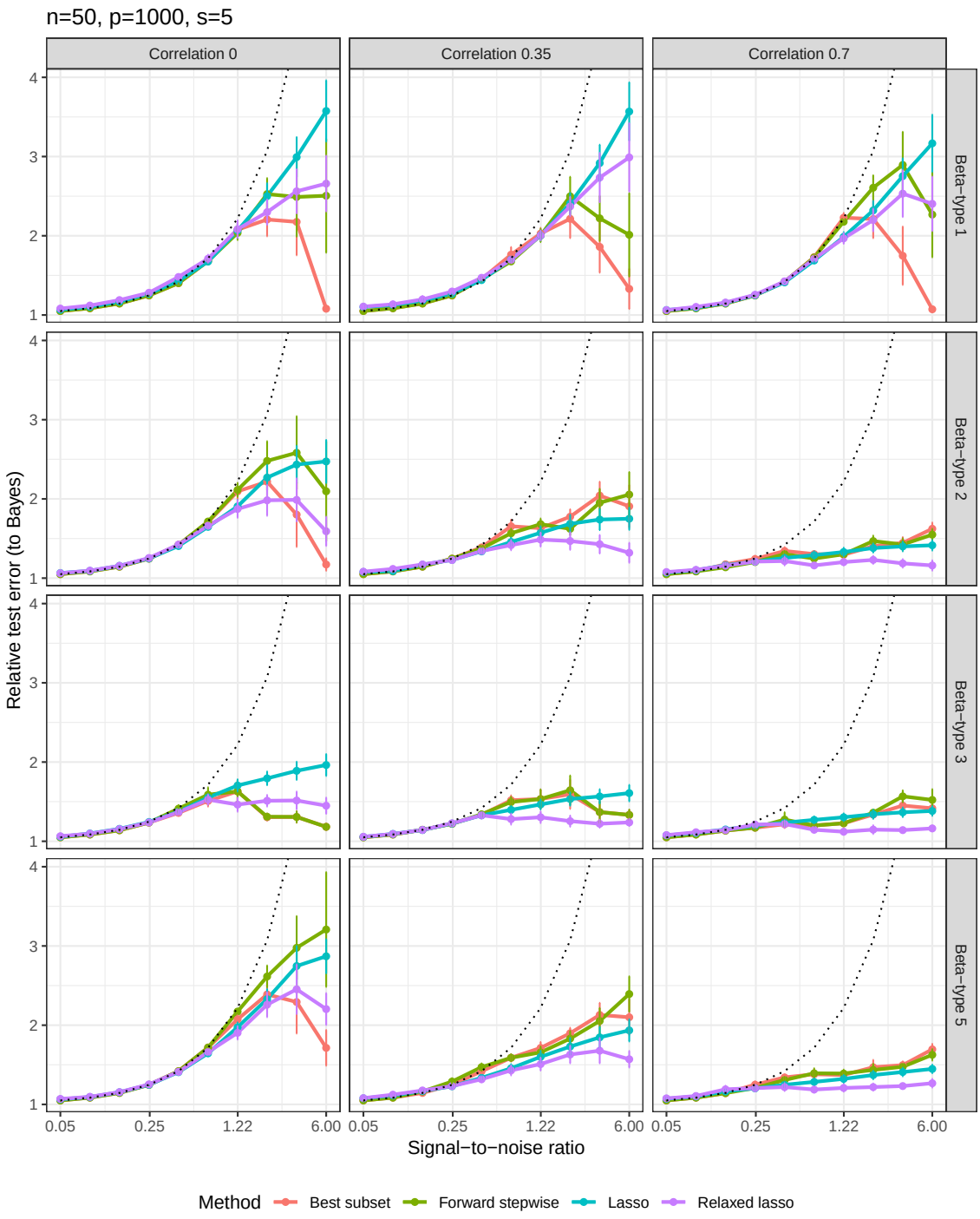

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

A.3 High-5 setting: $n = 50$, $p = 1000$, $s = 5$

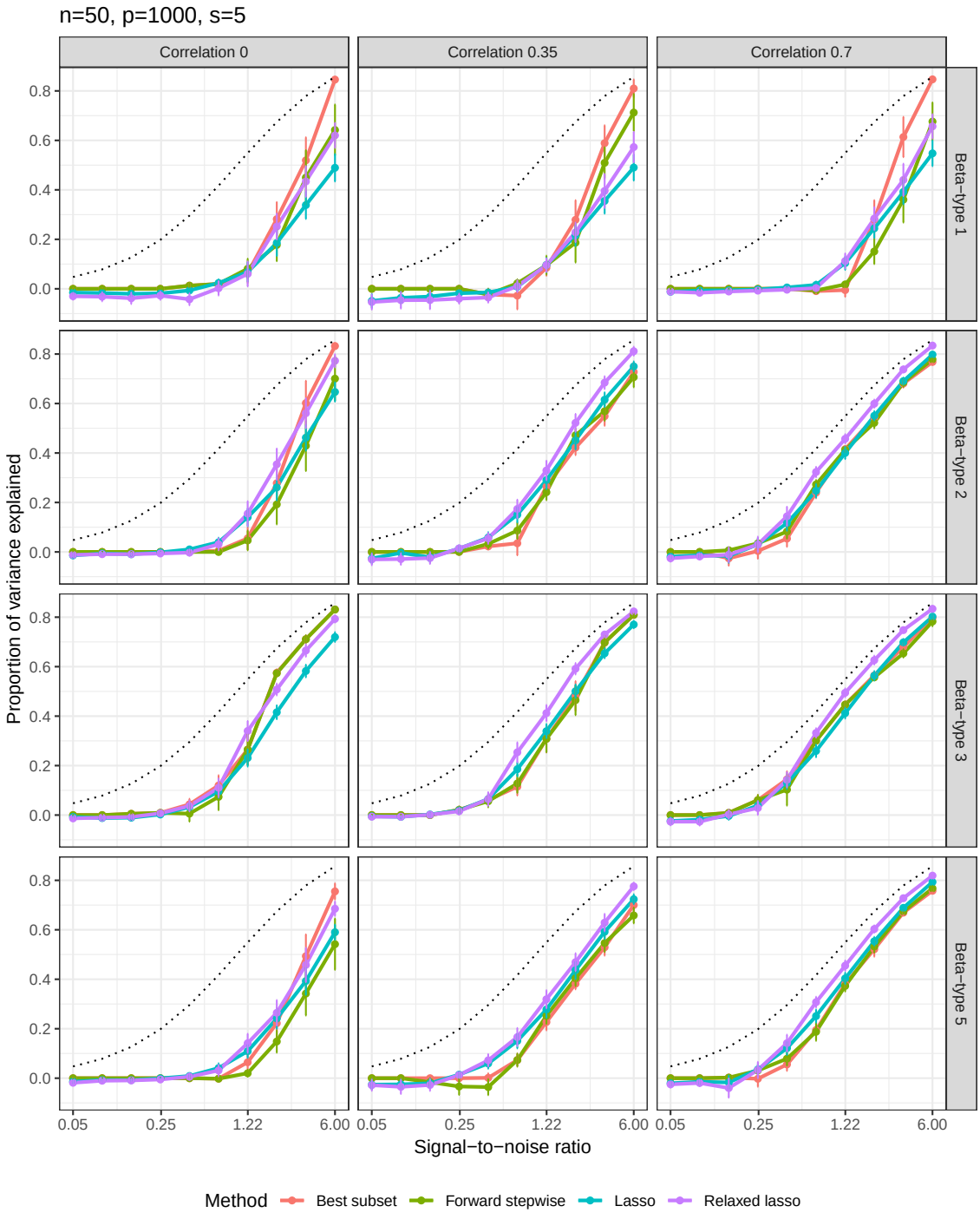
A.3.1 Relative risk (to null model)



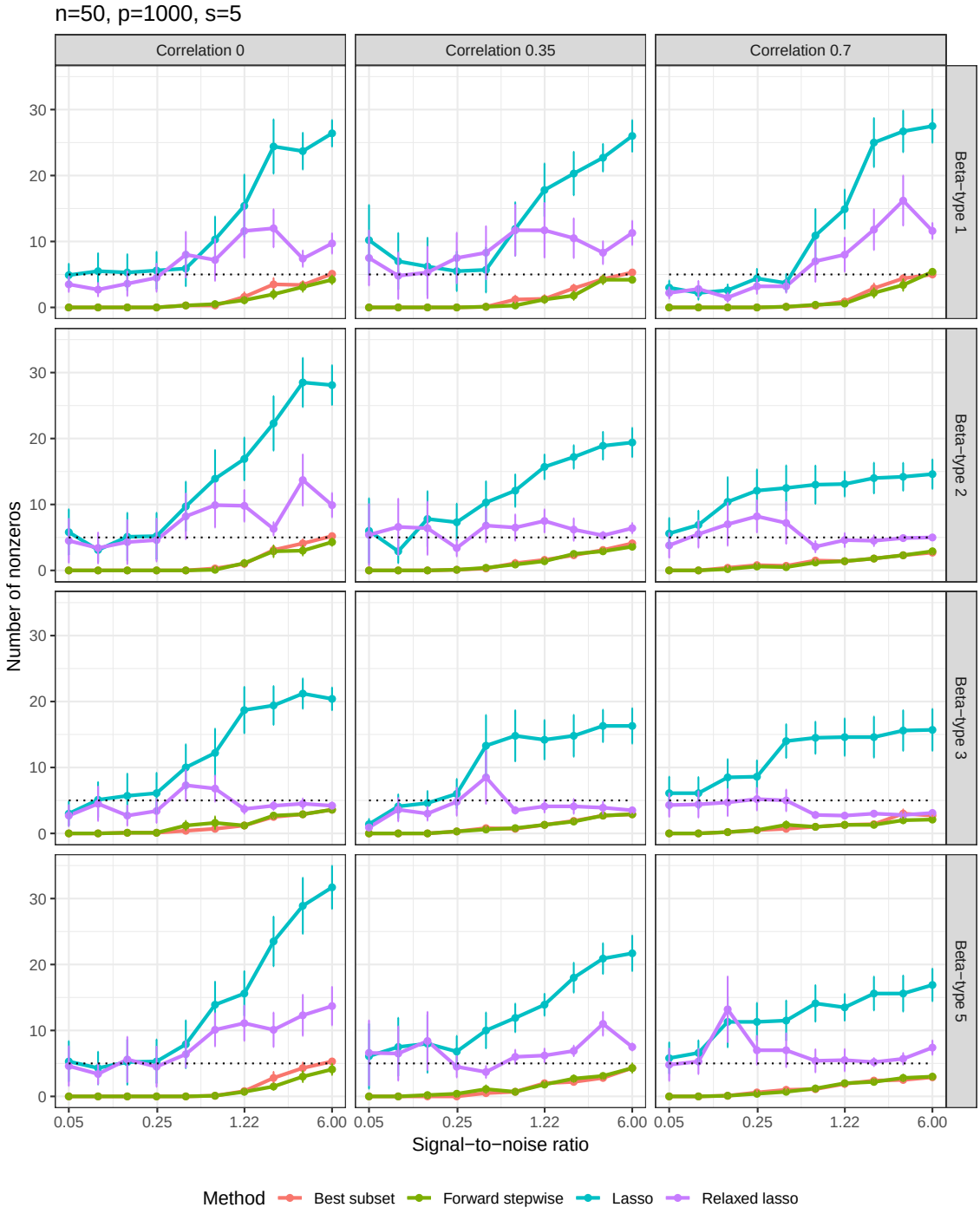
A.3.2 Relative test error (to Bayes)



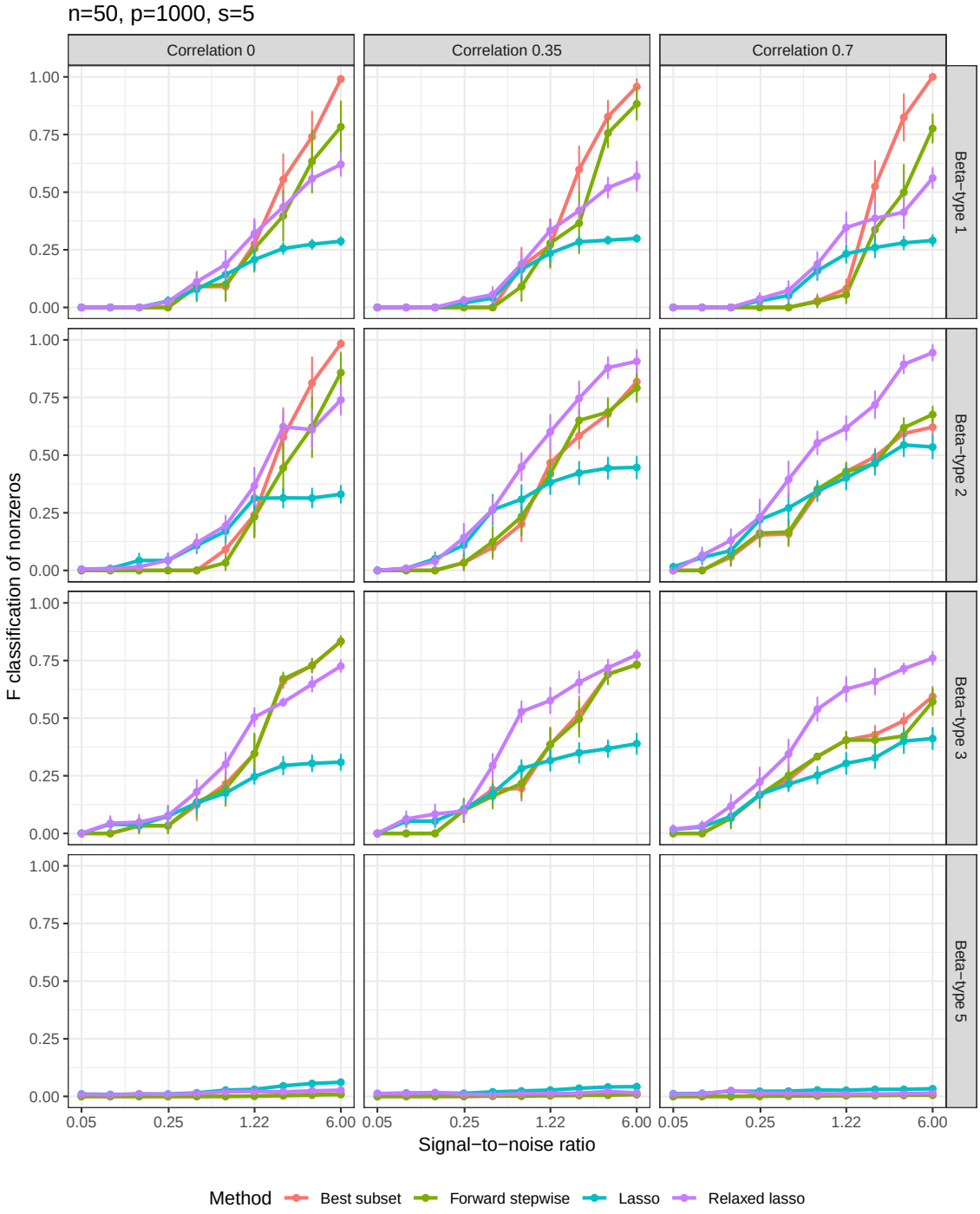
A.3.3 Proportion of variance explained



A.3.4 Number of nonzero coefficients

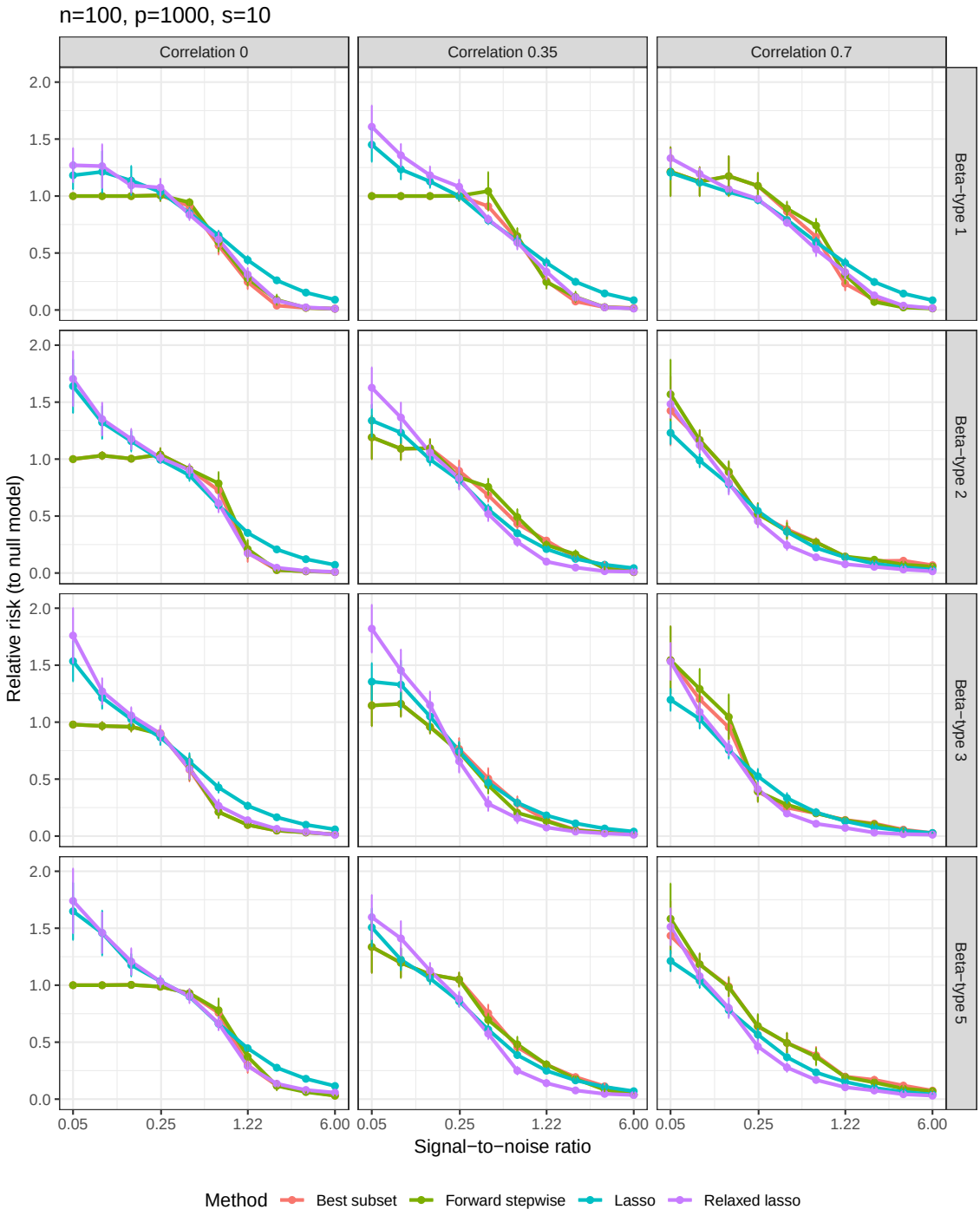


A.3.5 F-score

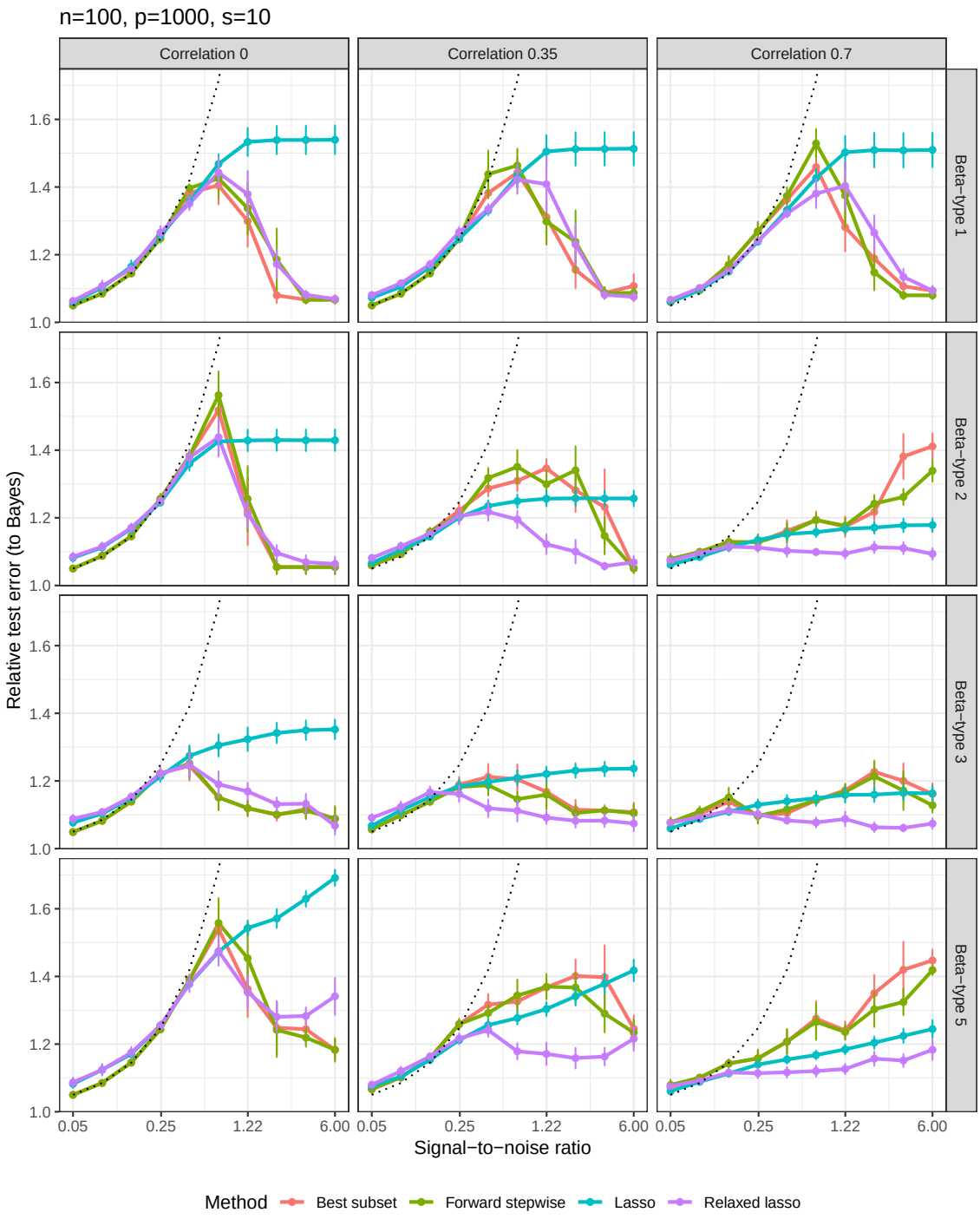

n=50, p=1000, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

A.4 High-10 setting: $n = 100$, $p = 1000$, $s = 10$

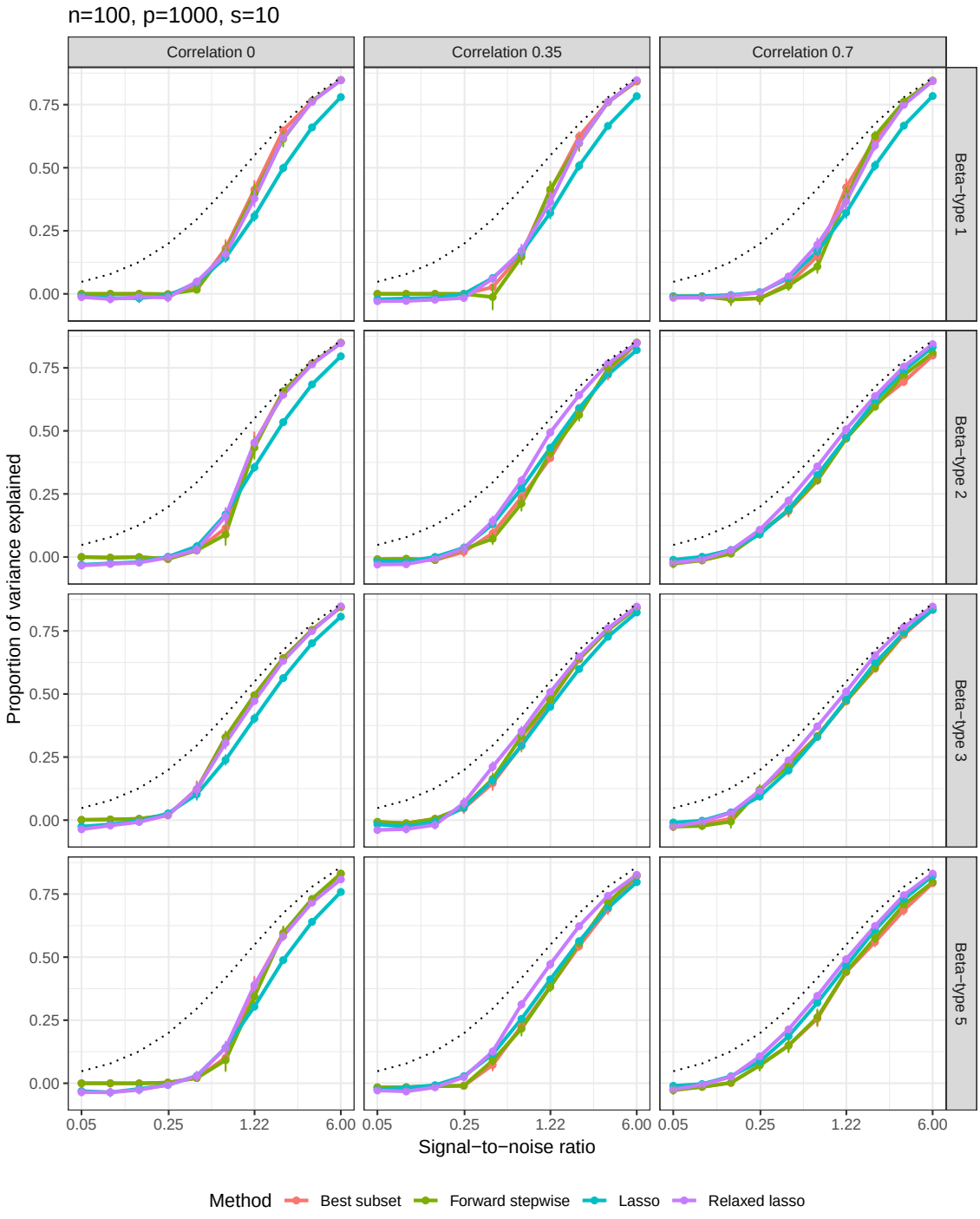
A.4.1 Relative risk (to null model)



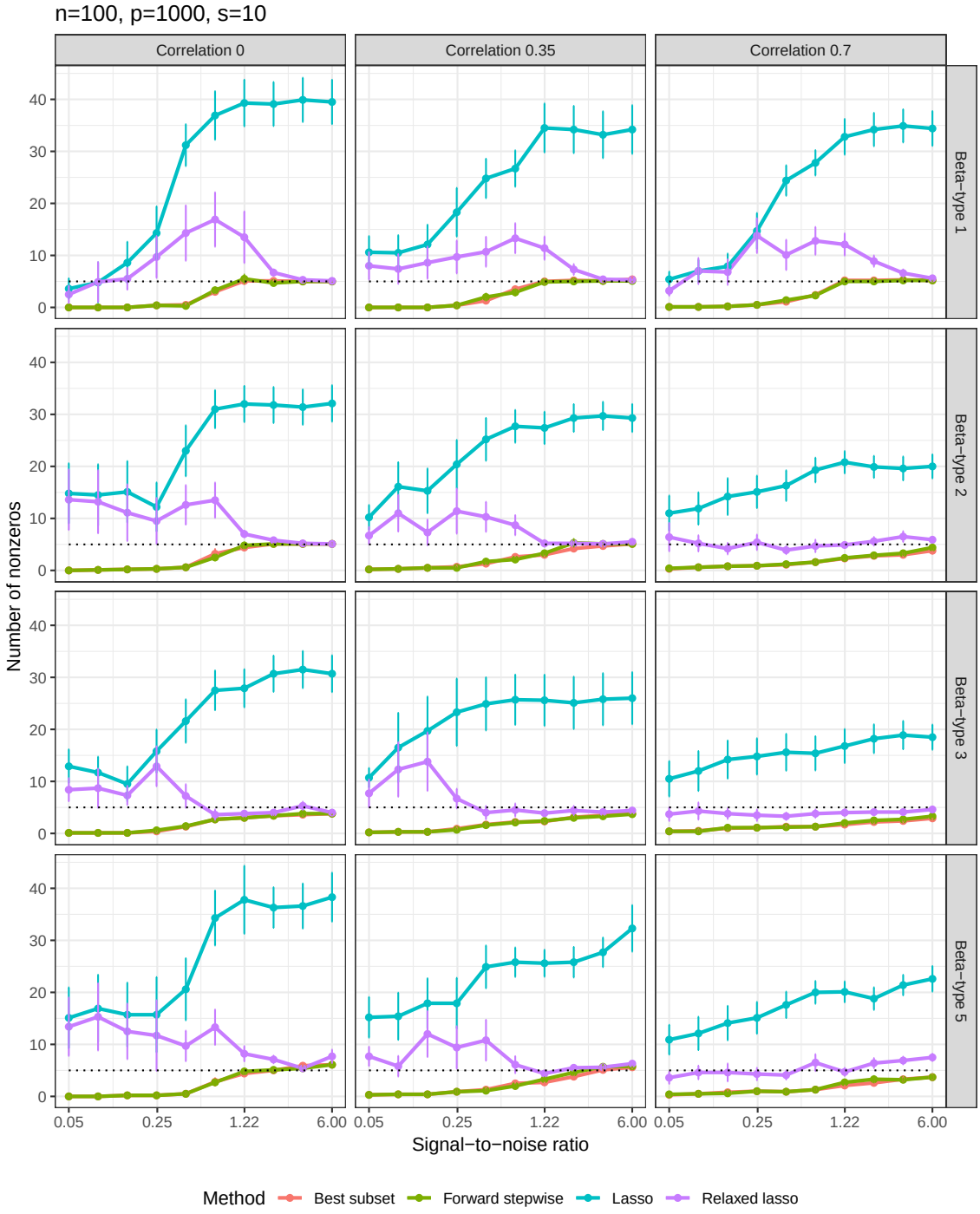
A.4.2 Relative test error (to Bayes)



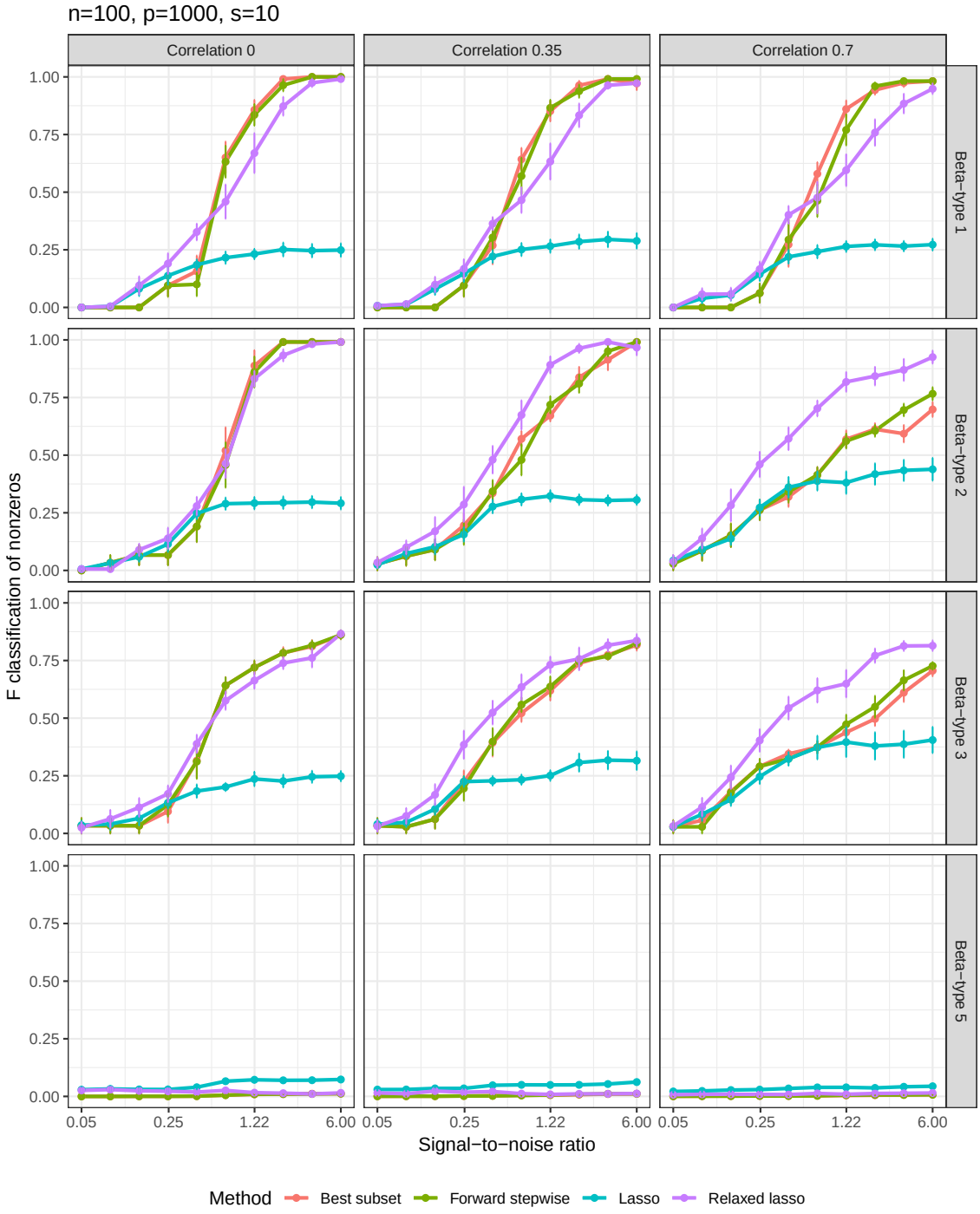
A.4.3 Proportion of variance explained



A.4.4 Number of nonzero coefficients



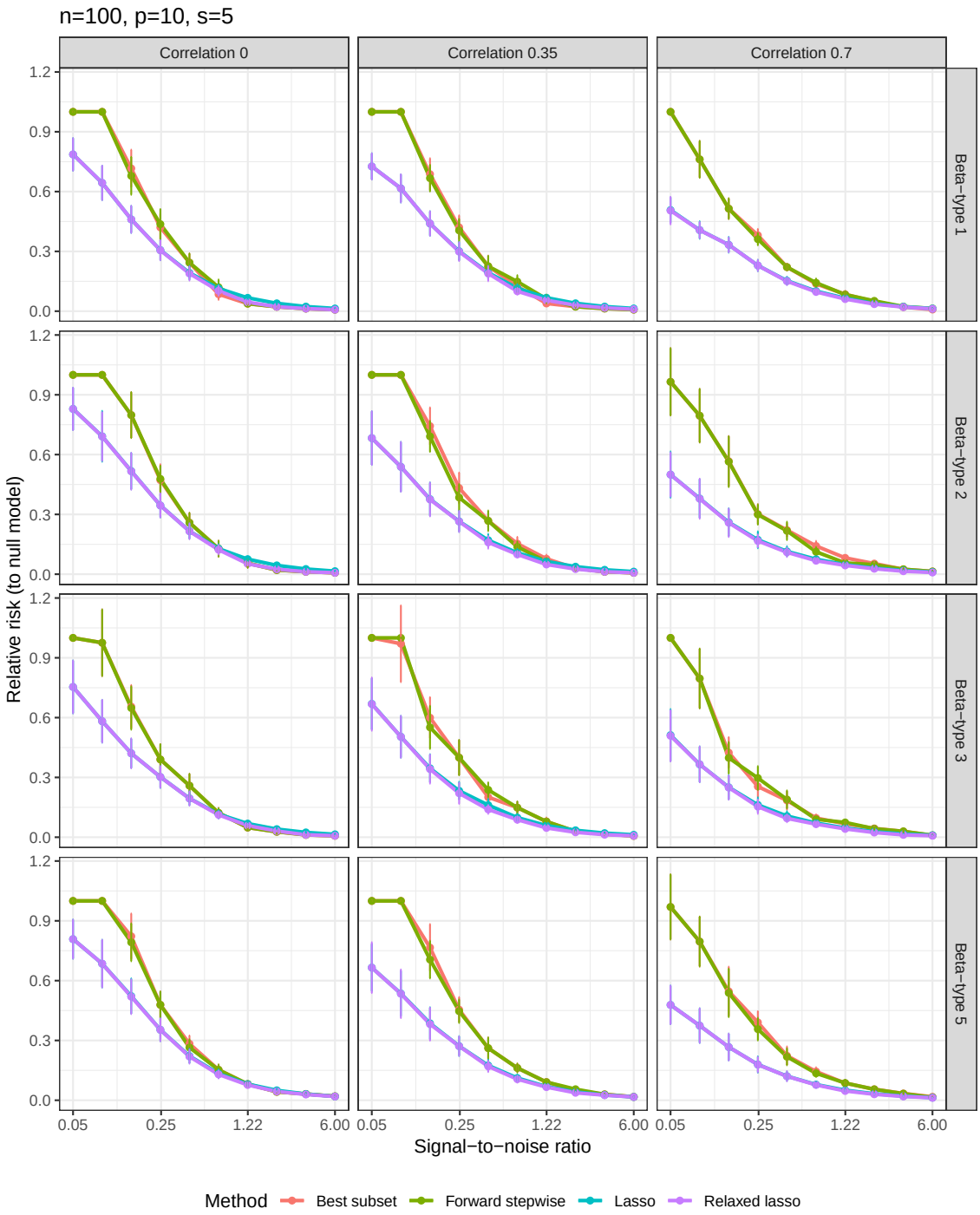
A.4.5 F-score


n=100, p=1000, s=10
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

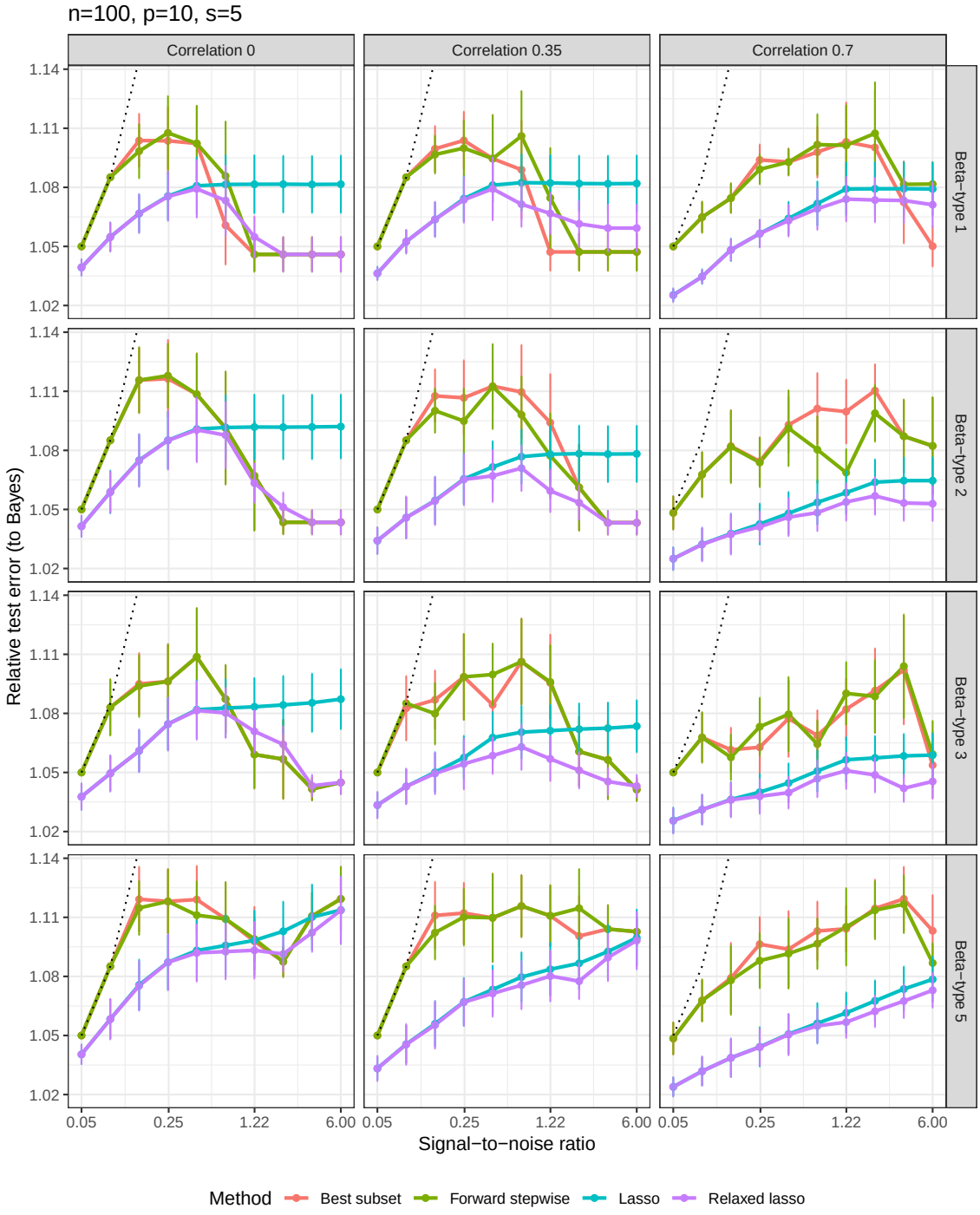
B Oracle tuning

B.1 Low setting: $n = 100$, $p = 10$, $s = 5$

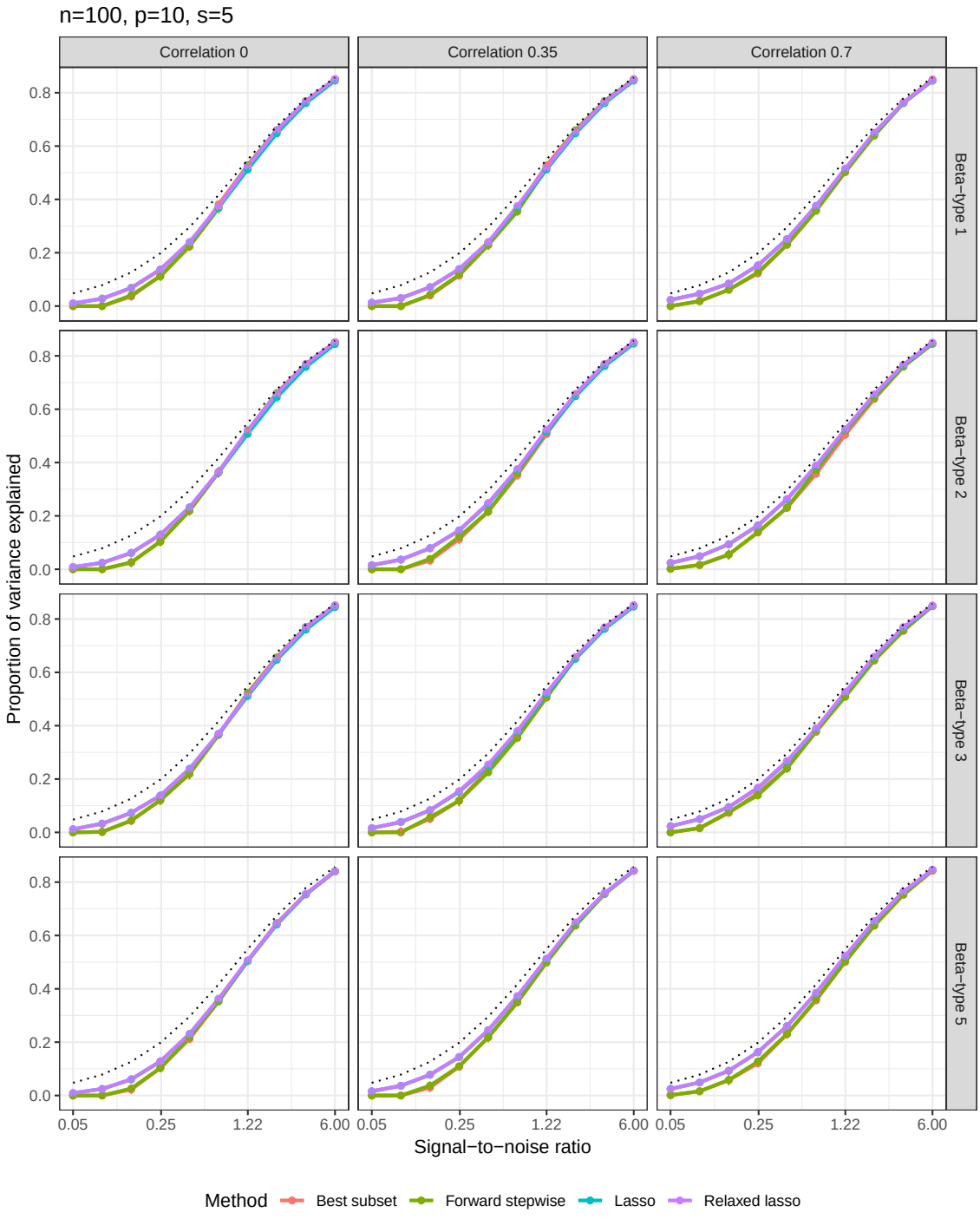
B.1.1 Relative risk (to null model)



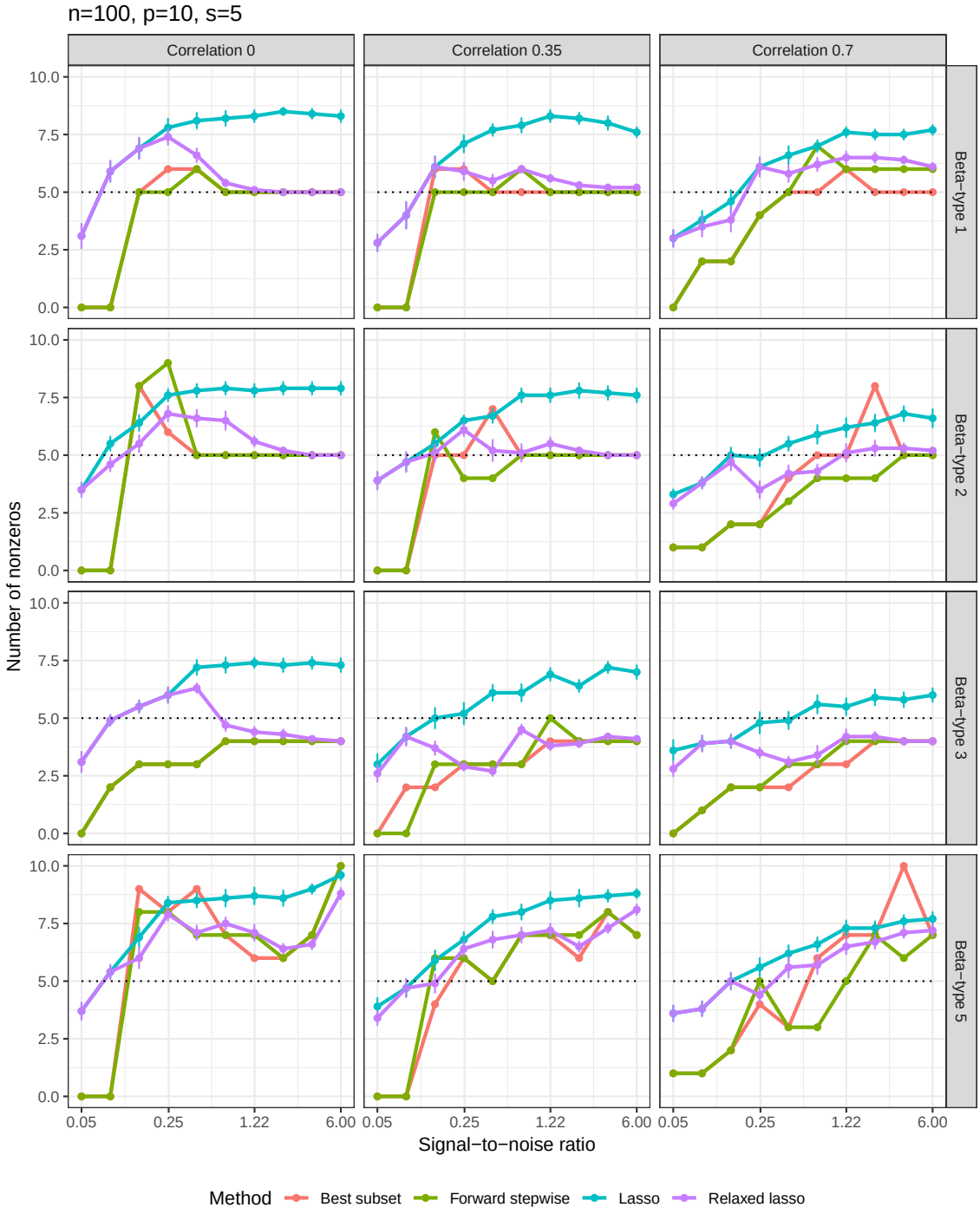
B.1.2 Relative test error (to Bayes)



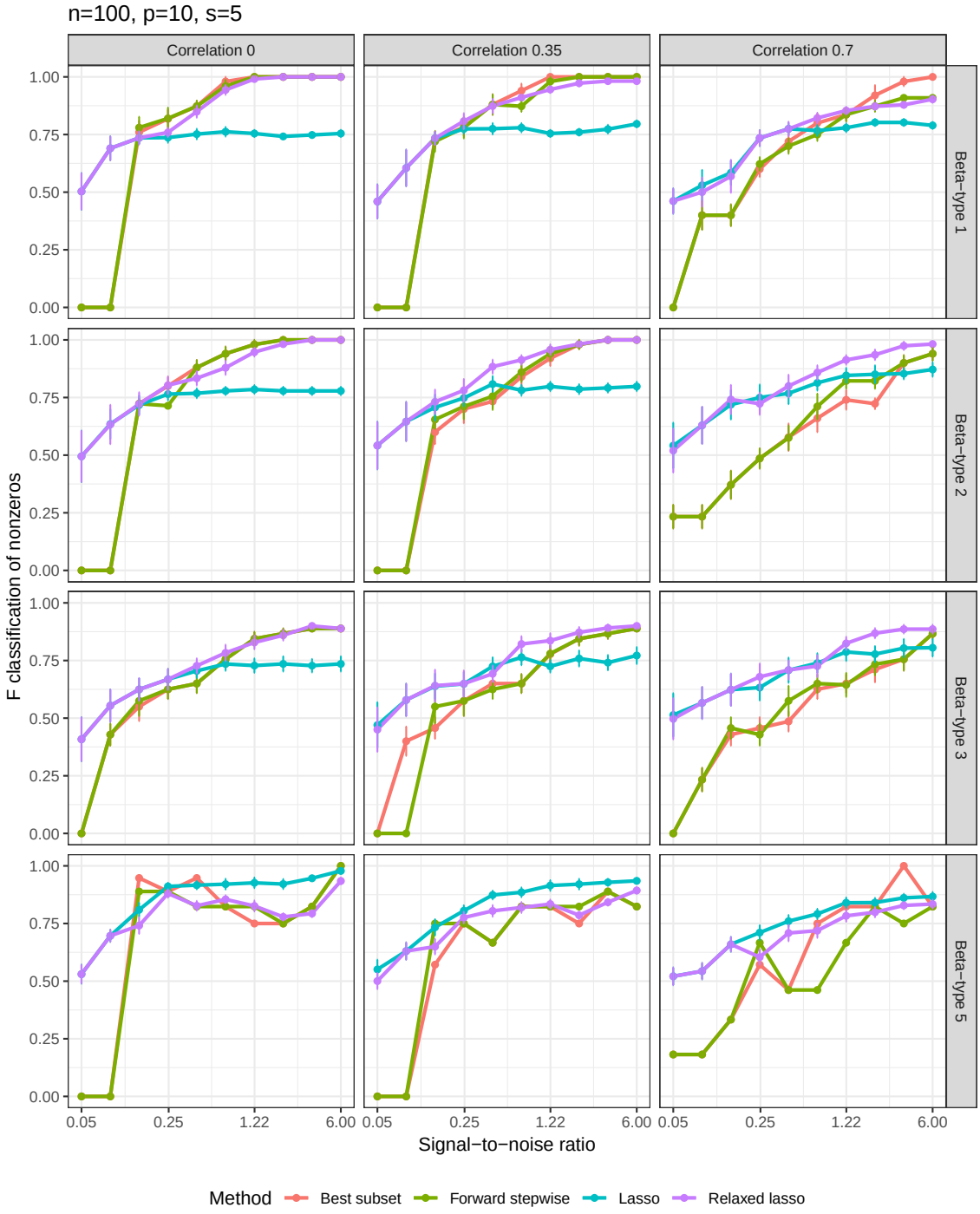
B.1.3 Proportion of variance explained



B.1.4 Number of nonzero coefficients

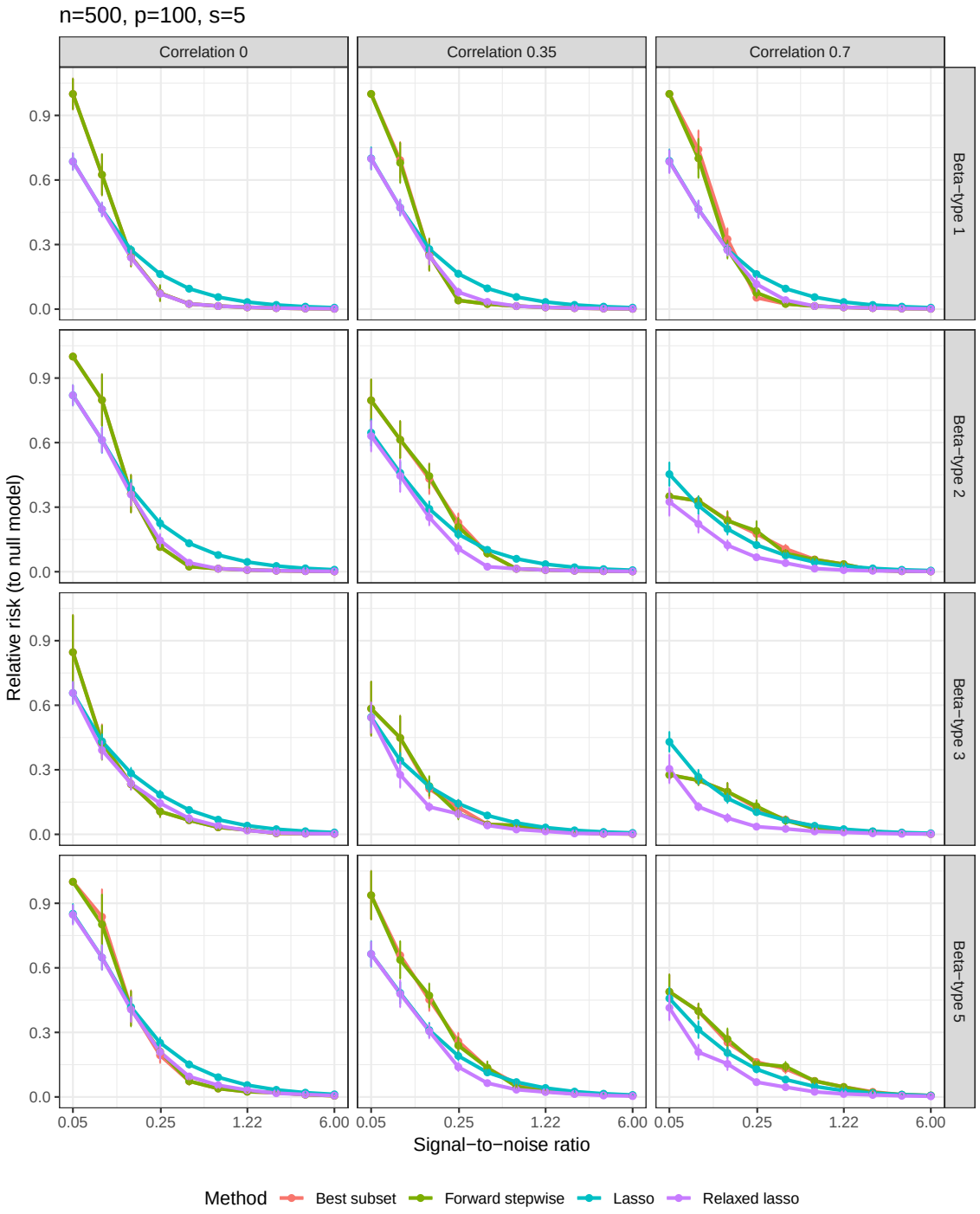


B.1.5 F-score

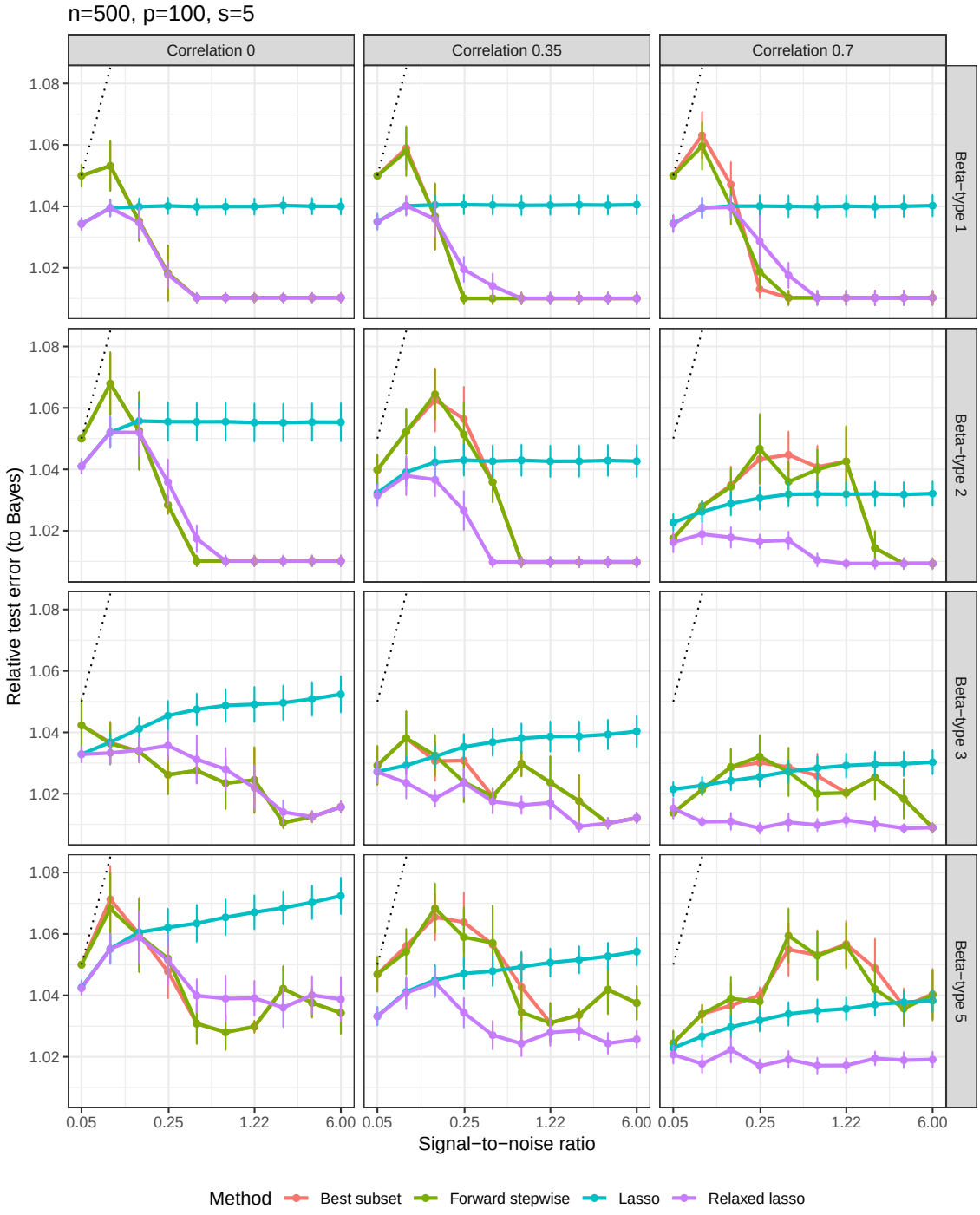

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

B.2 Medium setting: $n = 500$, $p = 100$, $s = 5$

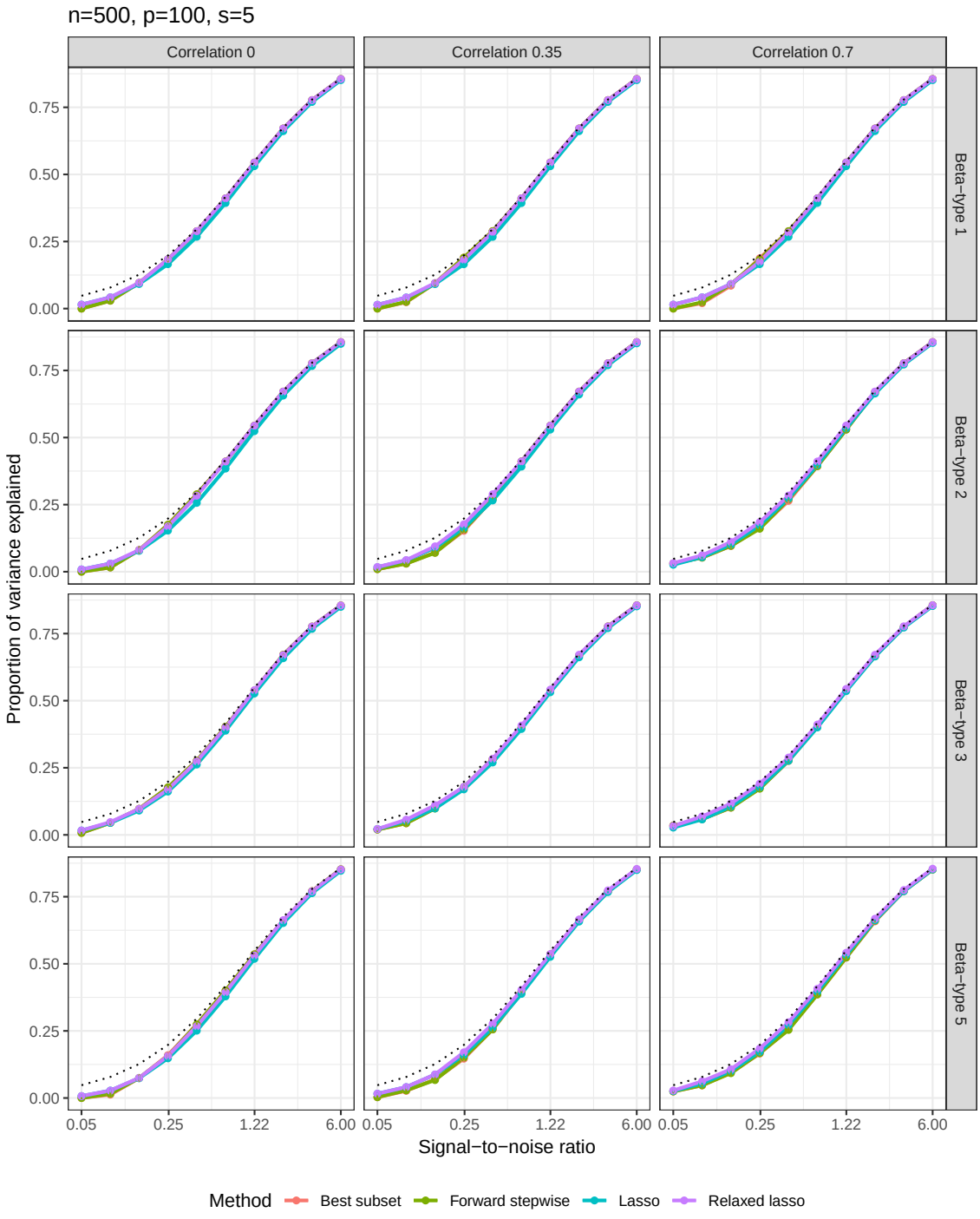
B.2.1 Relative risk (to null model)



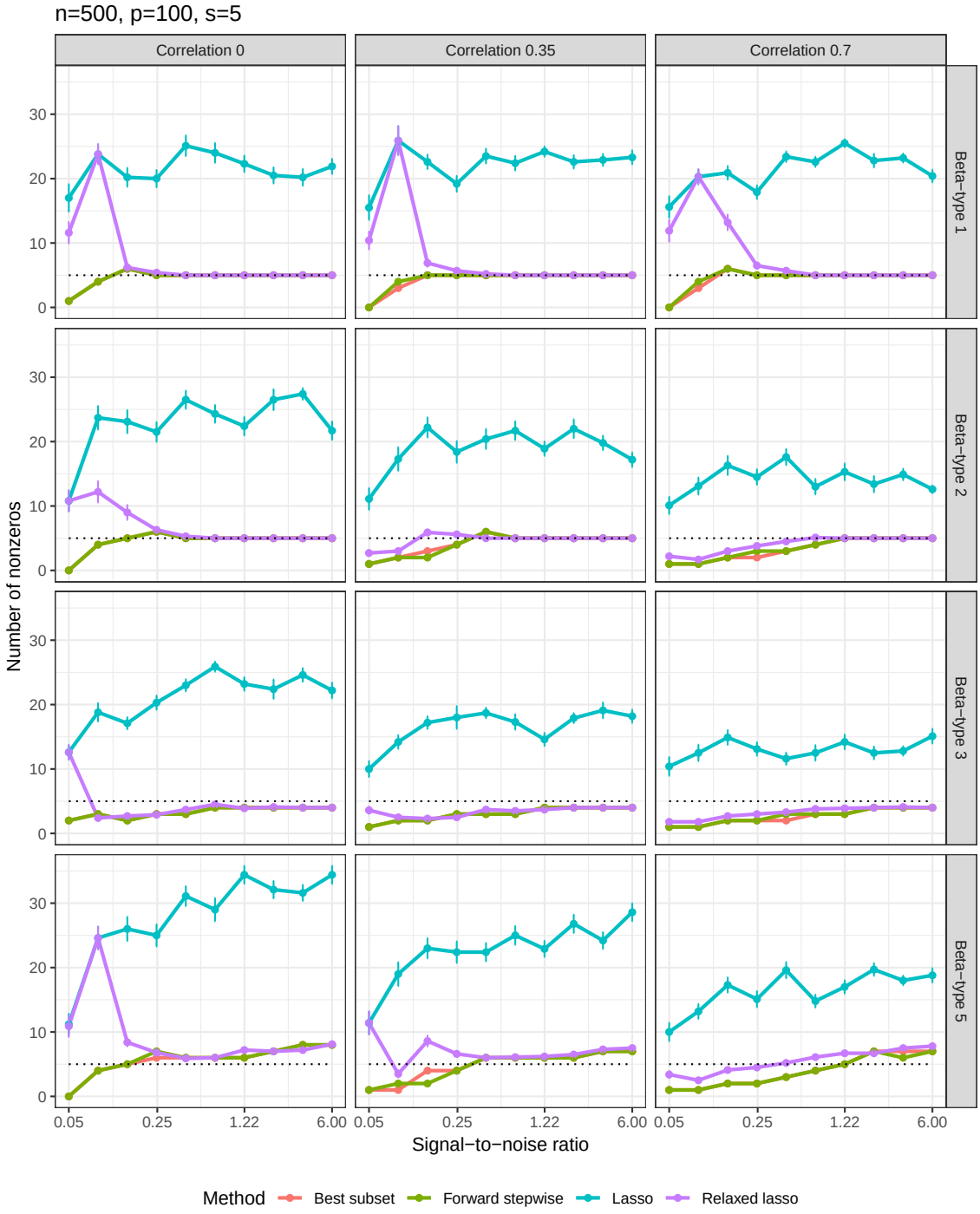
B.2.2 Relative test error (to Bayes)



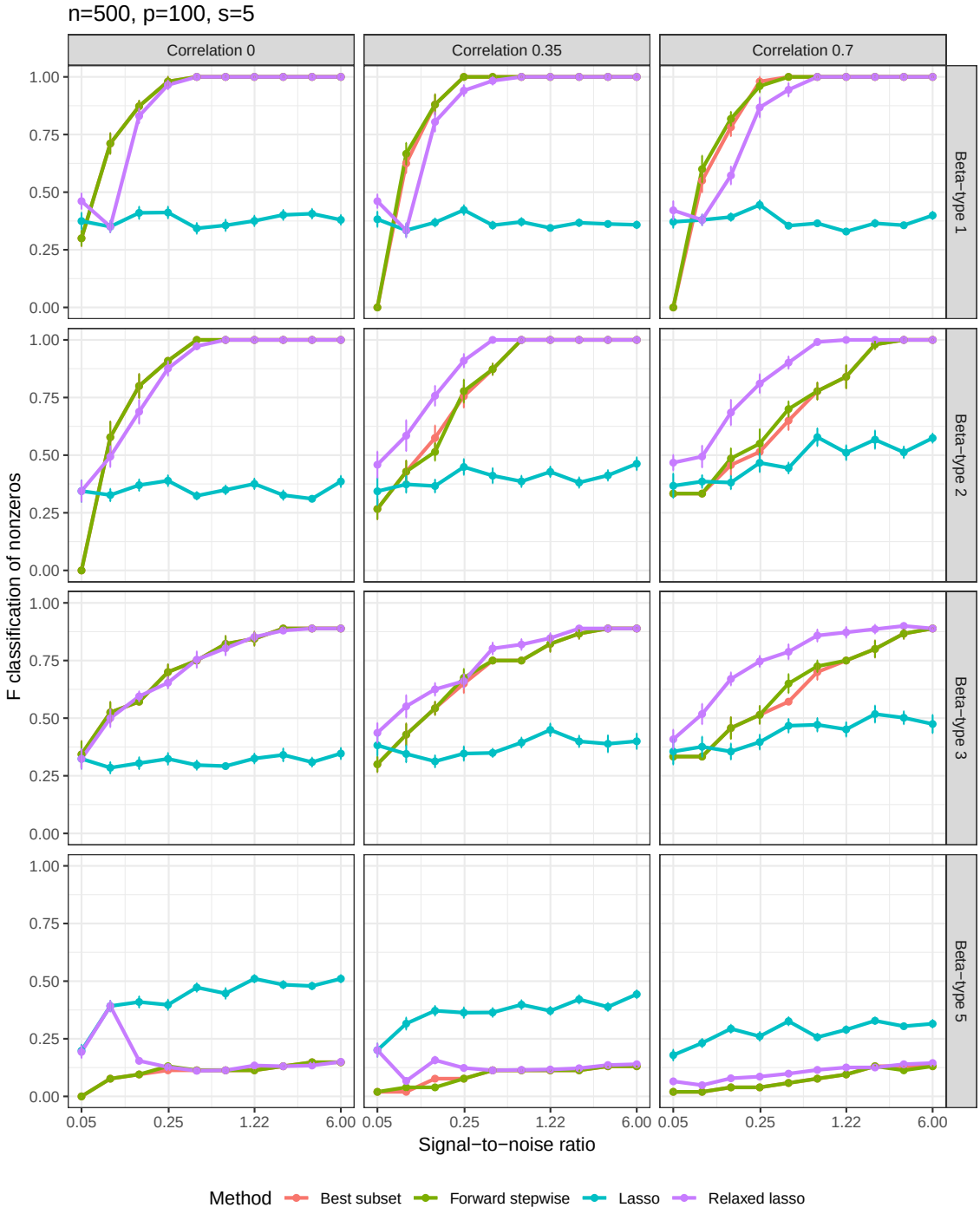
B.2.3 Proportion of variance explained



B.2.4 Number of nonzero coefficients

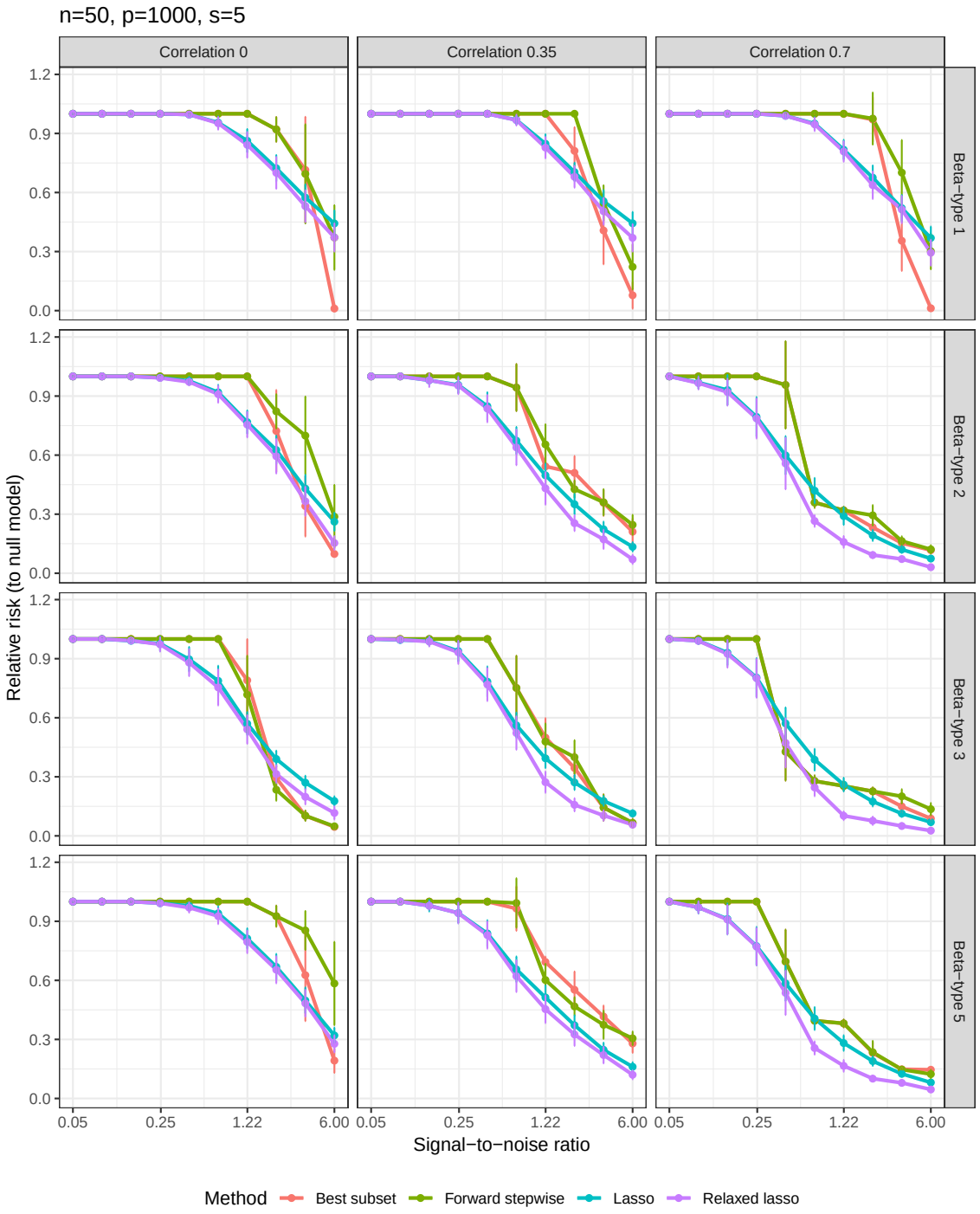


B.2.5 F-score

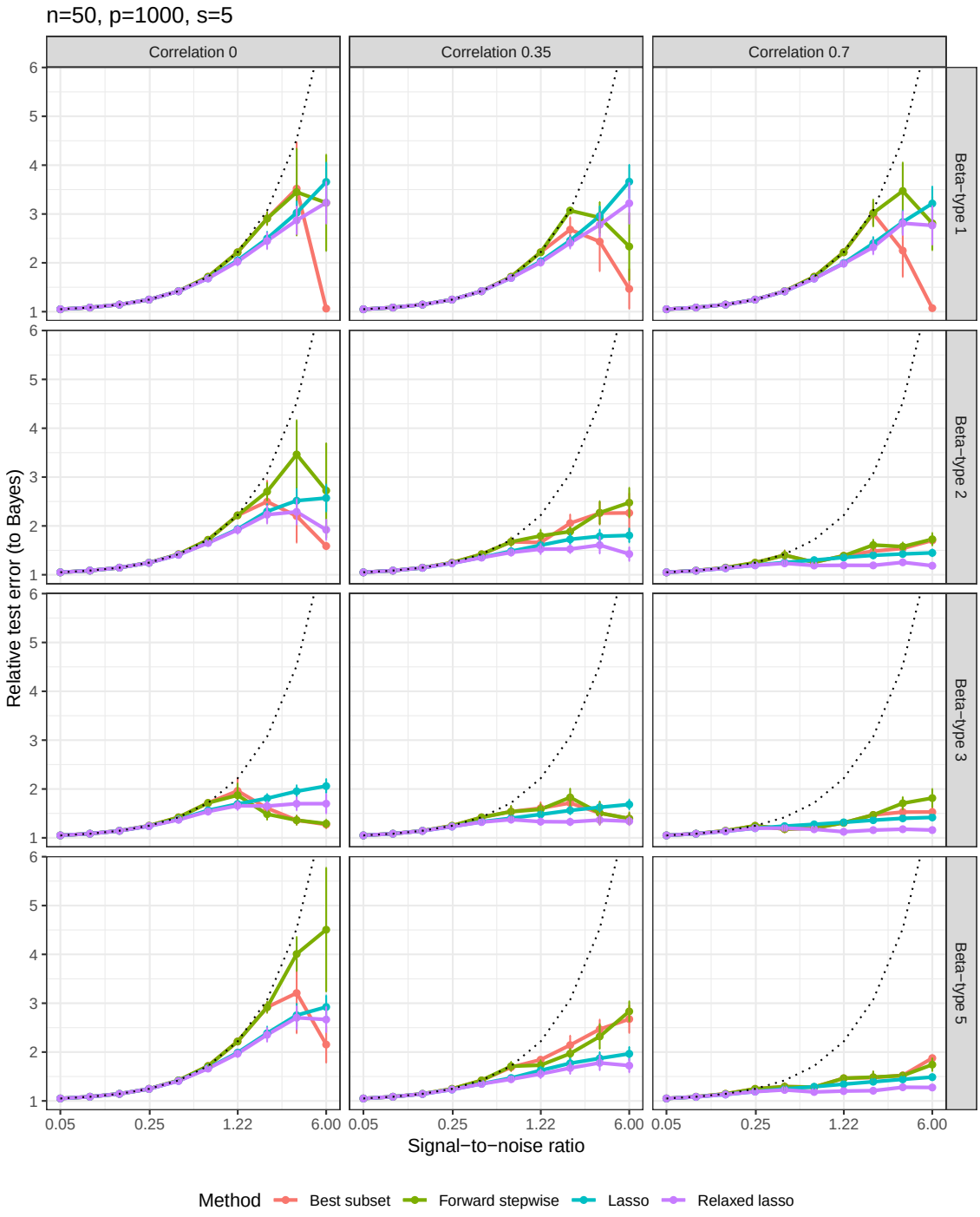

n=500, p=100, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

B.3 High-5 setting: $n = 50$, $p = 1000$, $s = 5$

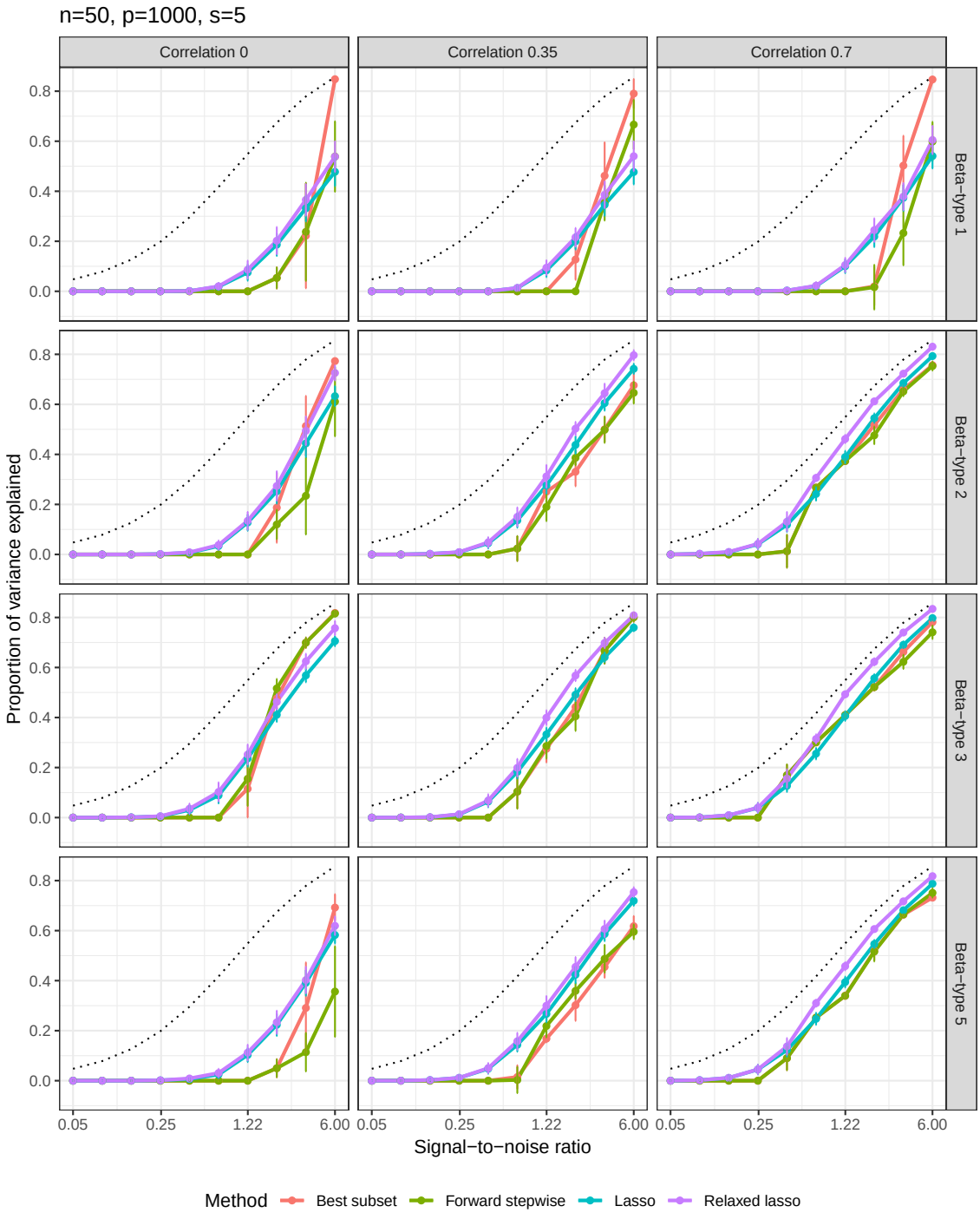
B.3.1 Relative risk (to null model)



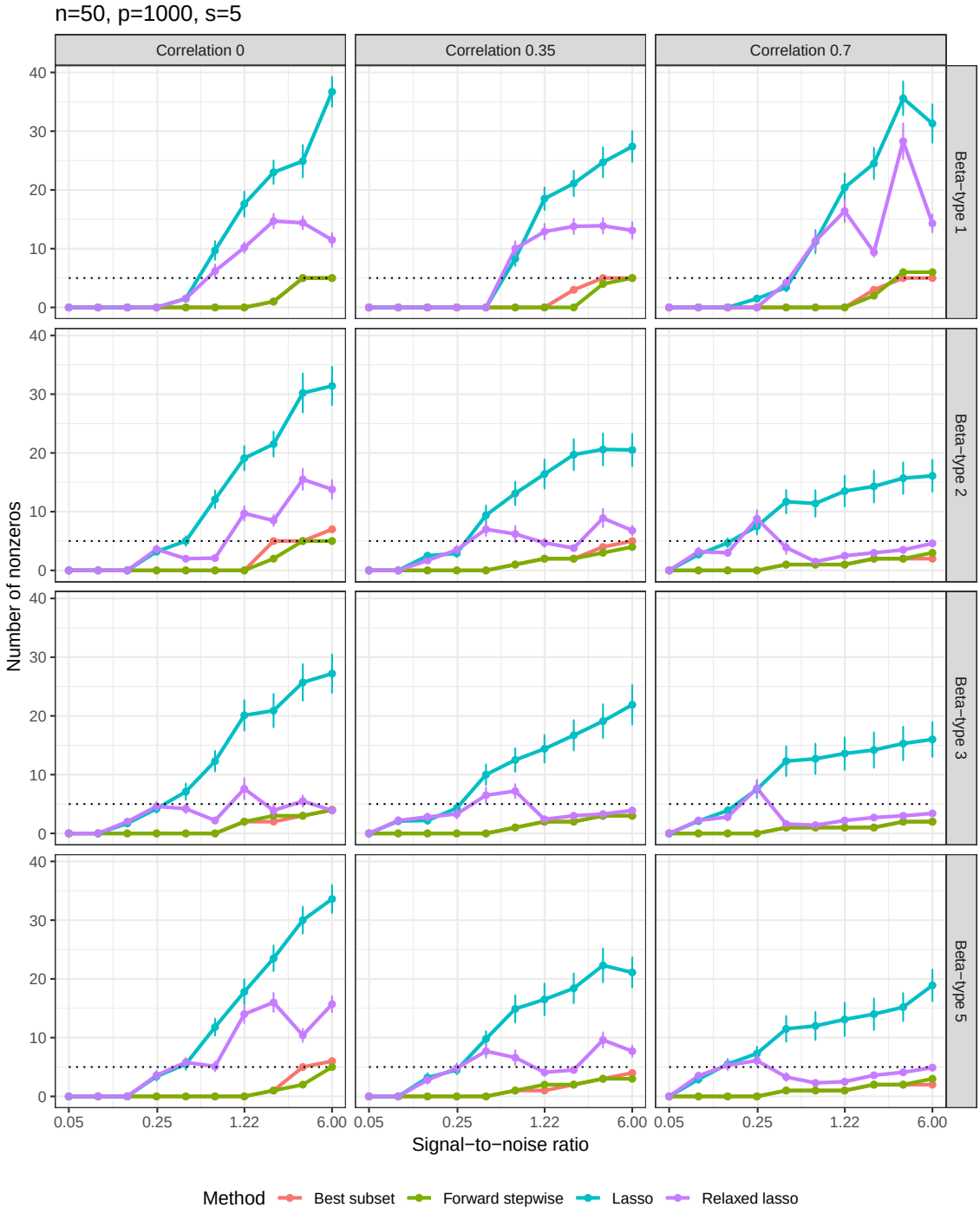
B.3.2 Relative test error (to Bayes)



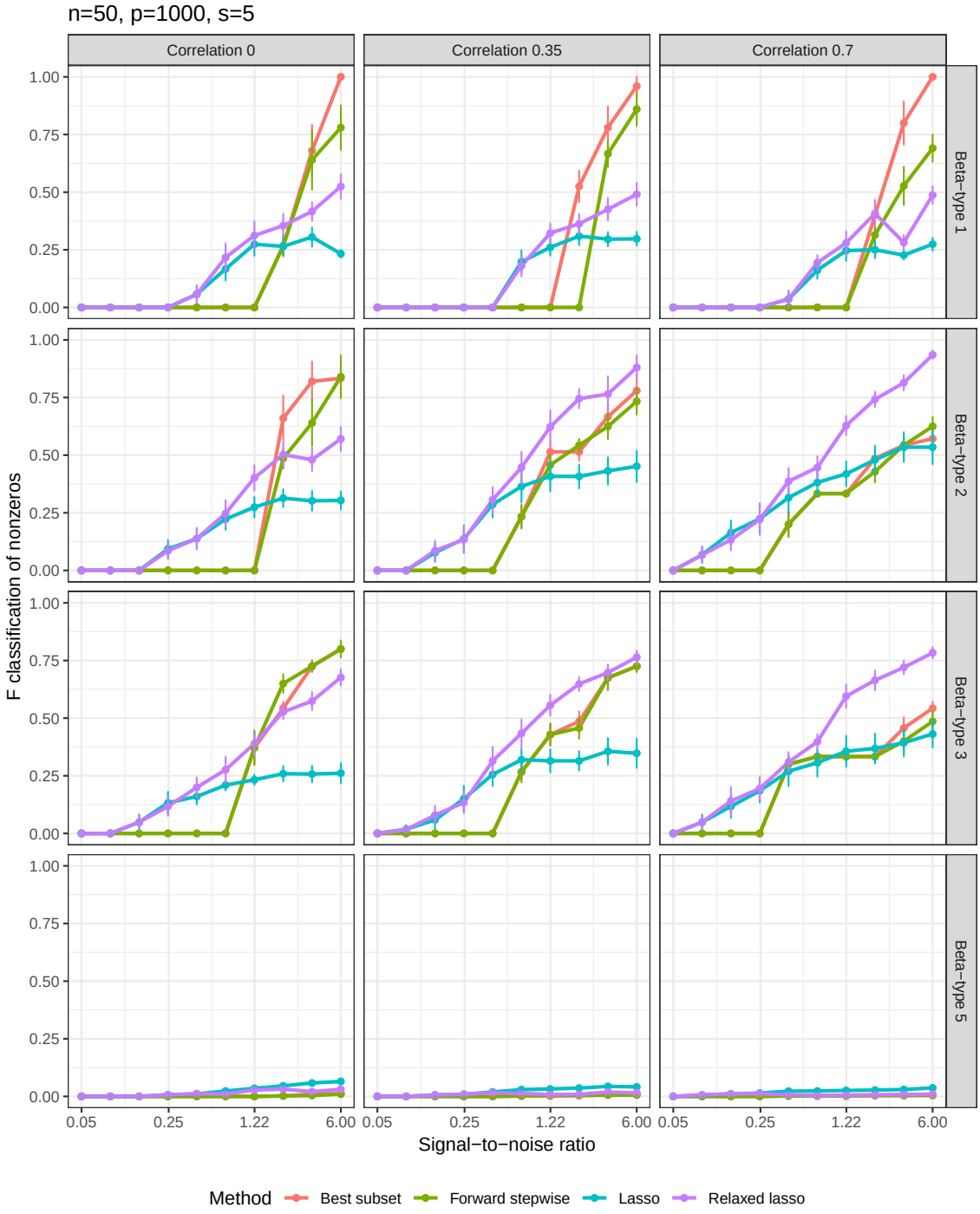
B.3.3 Proportion of variance explained



B.3.4 Number of nonzero coefficients

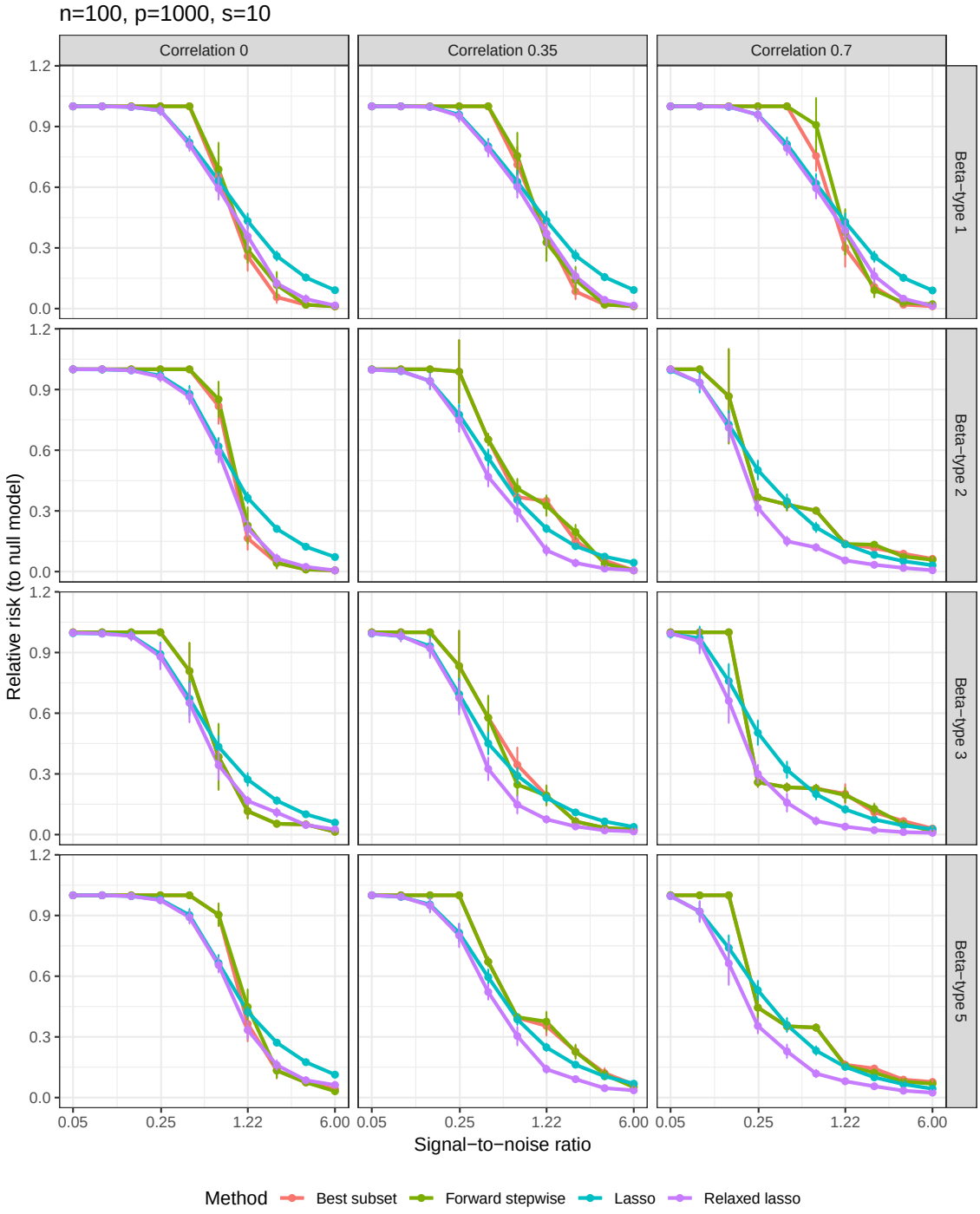


B.3.5 F-score

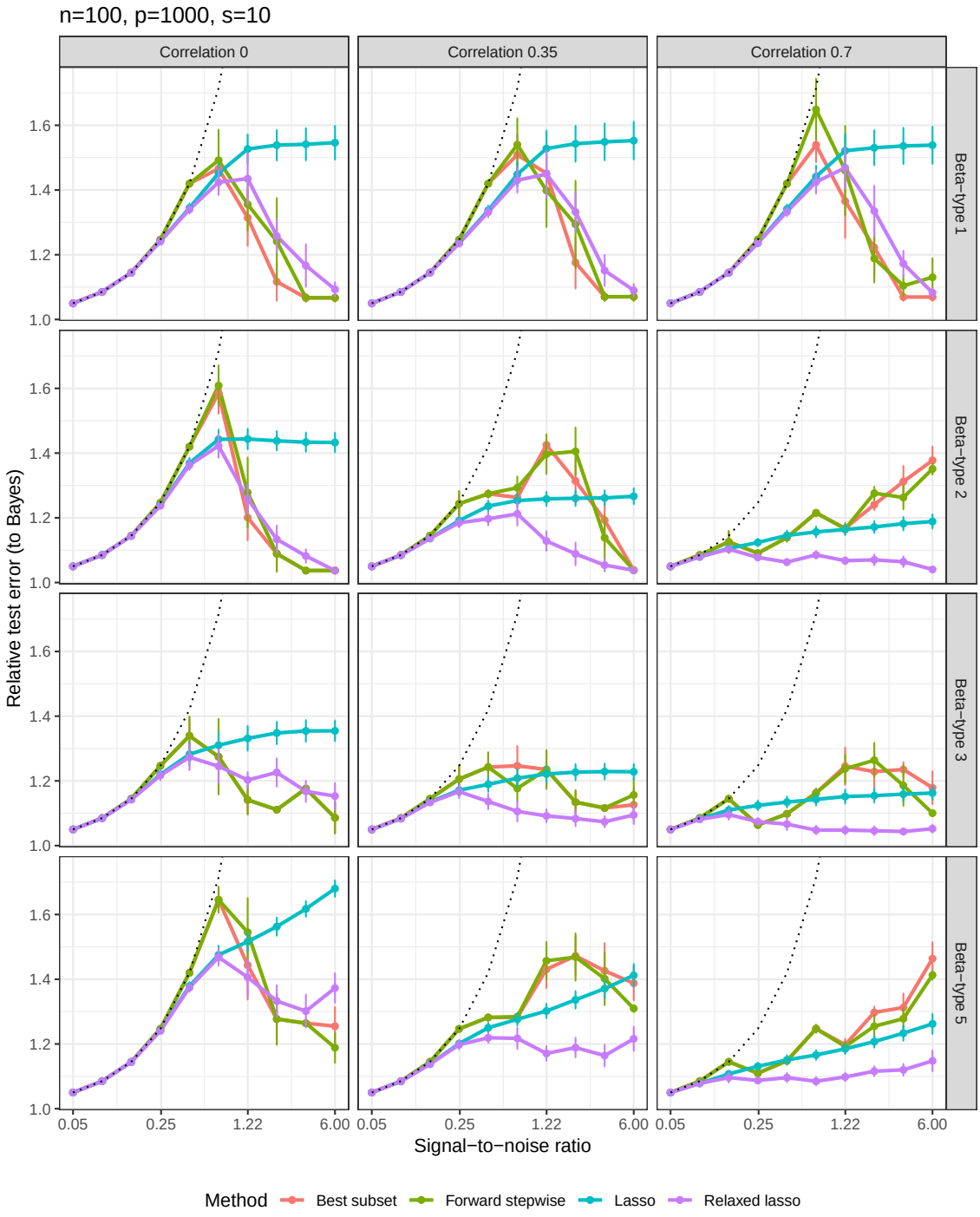

n=50, p=1000, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

B.4 High-10 setting: $n = 100$, $p = 1000$, $s = 10$

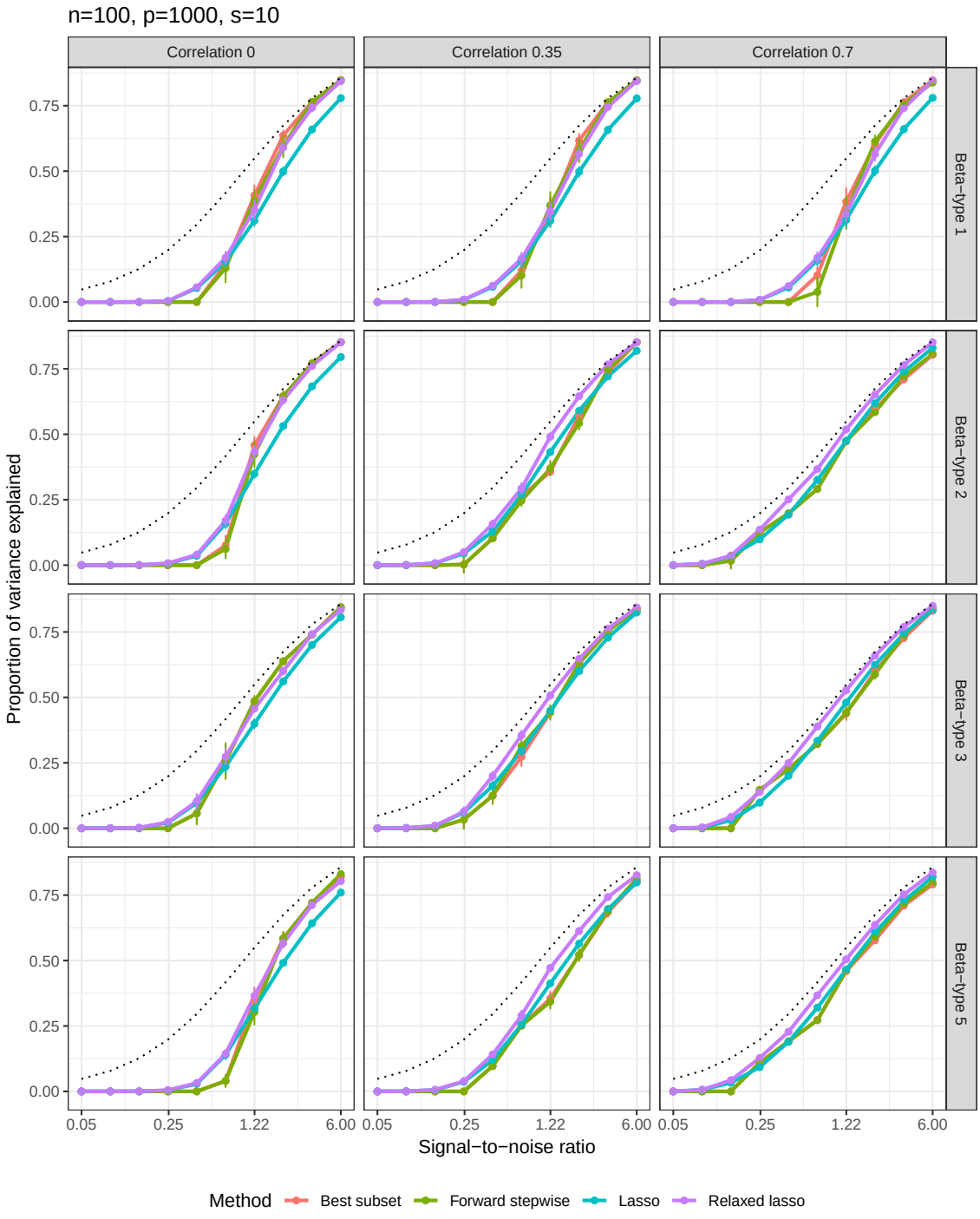
B.4.1 Relative risk (to null model)



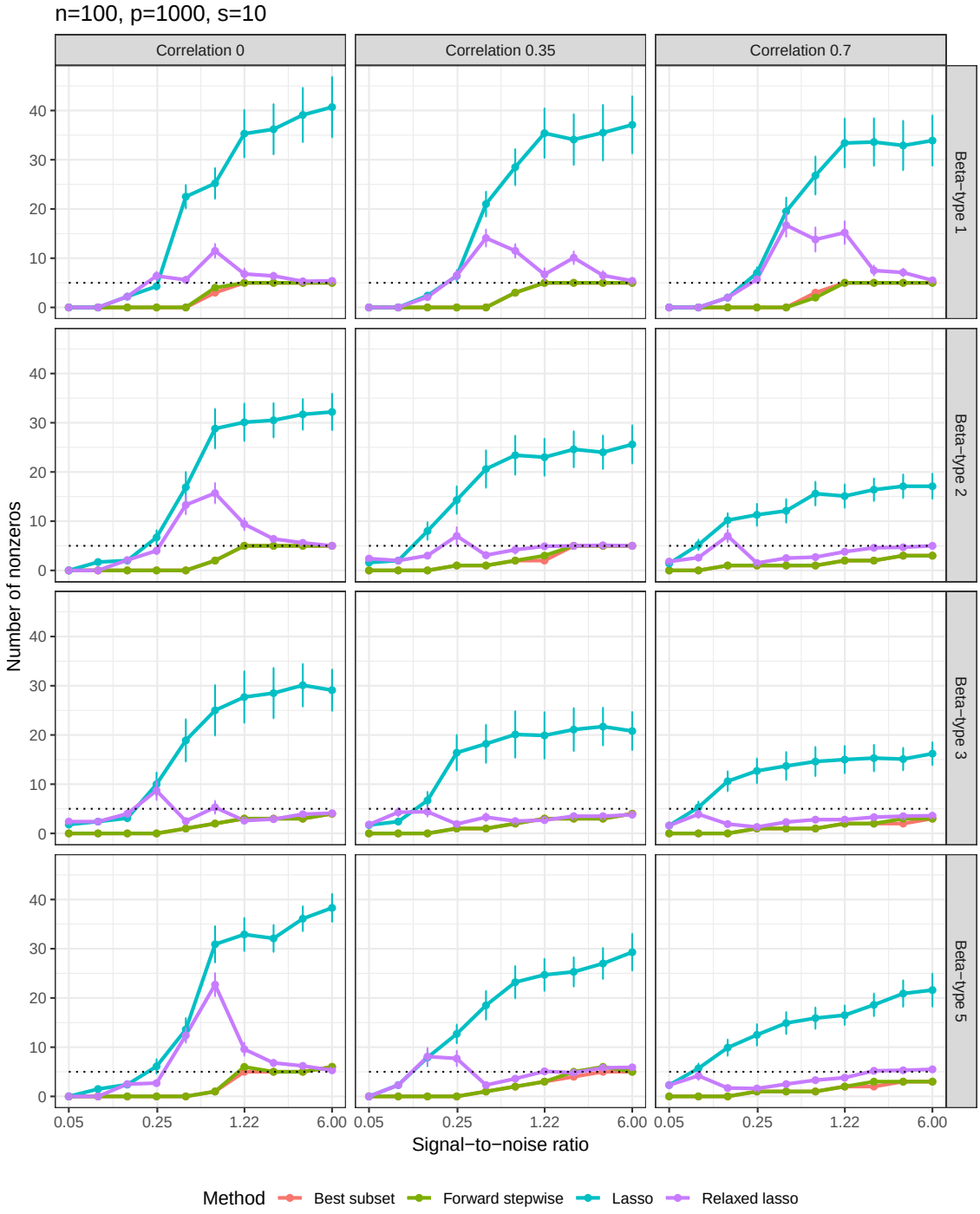
B.4.2 Relative test error (to Bayes)



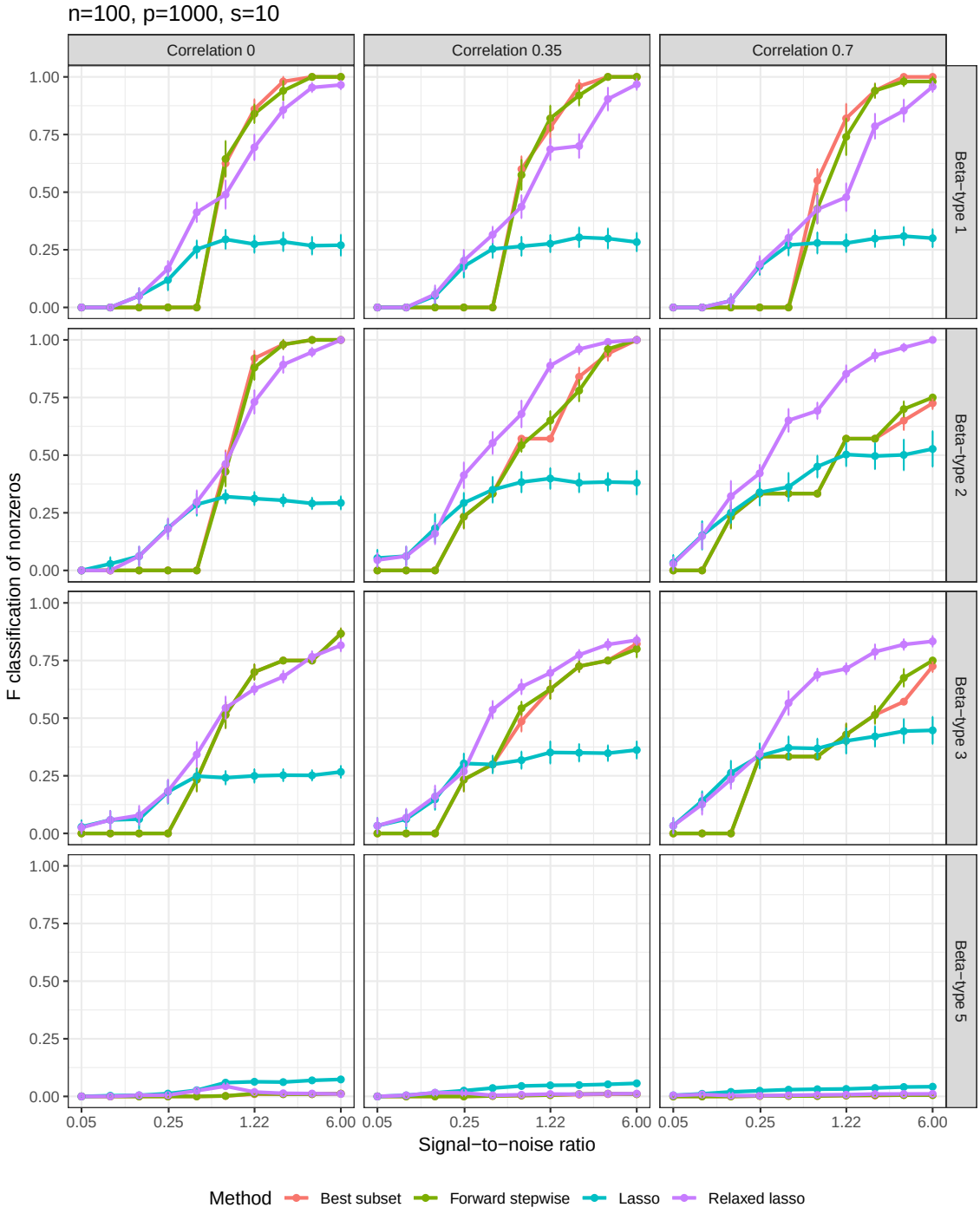
B.4.3 Proportion of variance explained



B.4.4 Number of nonzero coefficients



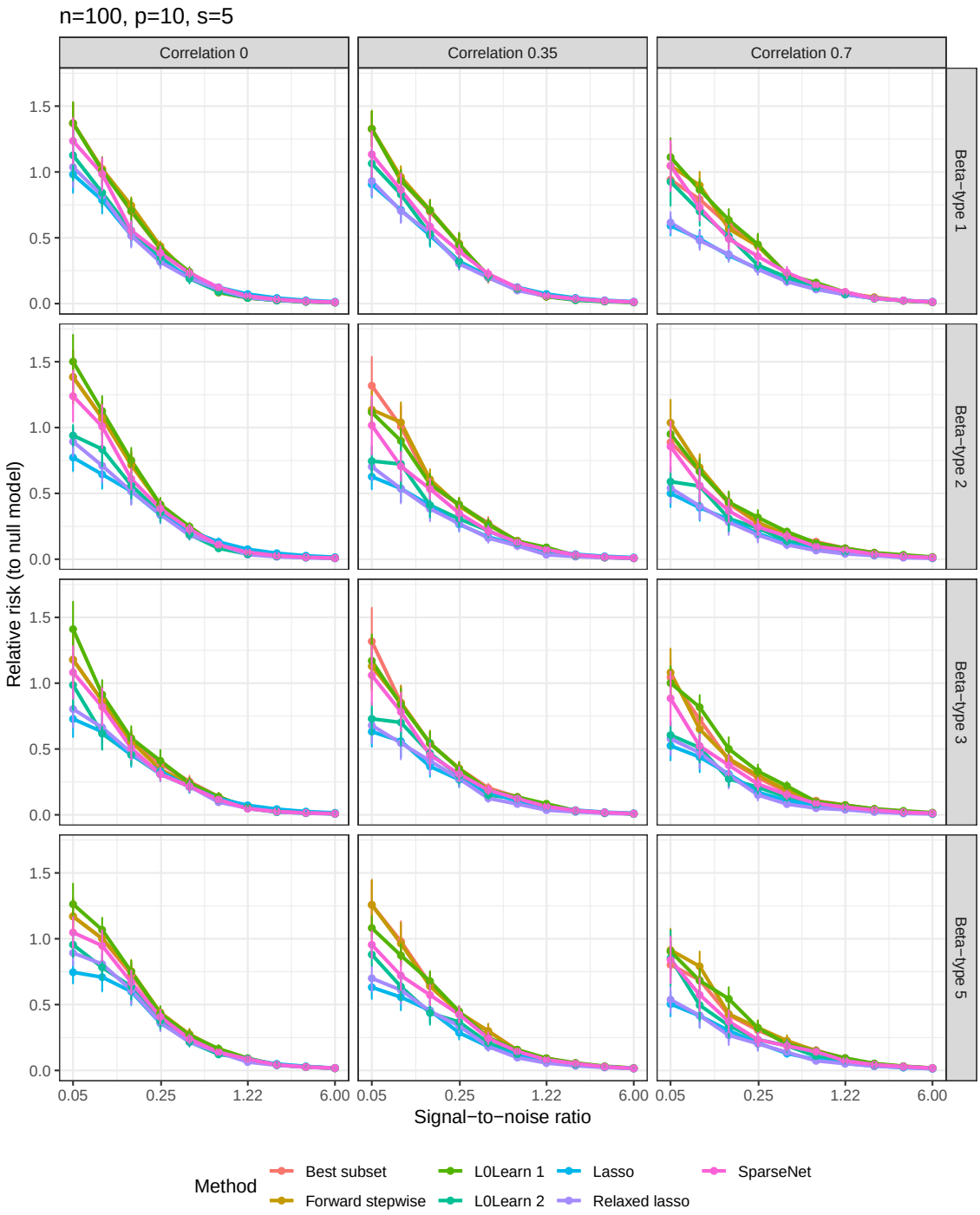
B.4.5 F-score


n=100, p=1000, s=10
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
Lasso
Relaxed lasso

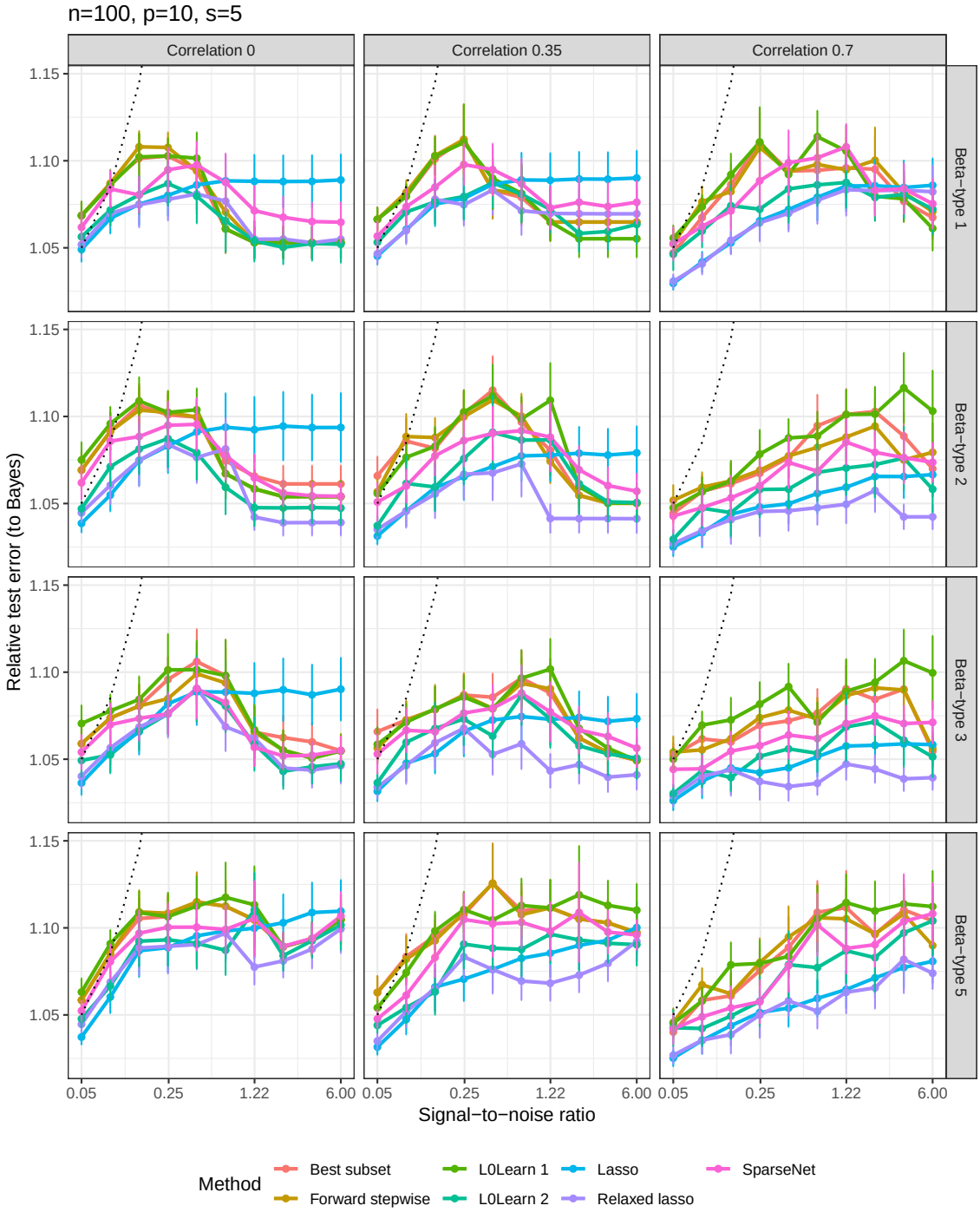
C Validation tuning: more methods

C.1 Low setting: $n = 100$, $p = 10$, $s = 5$

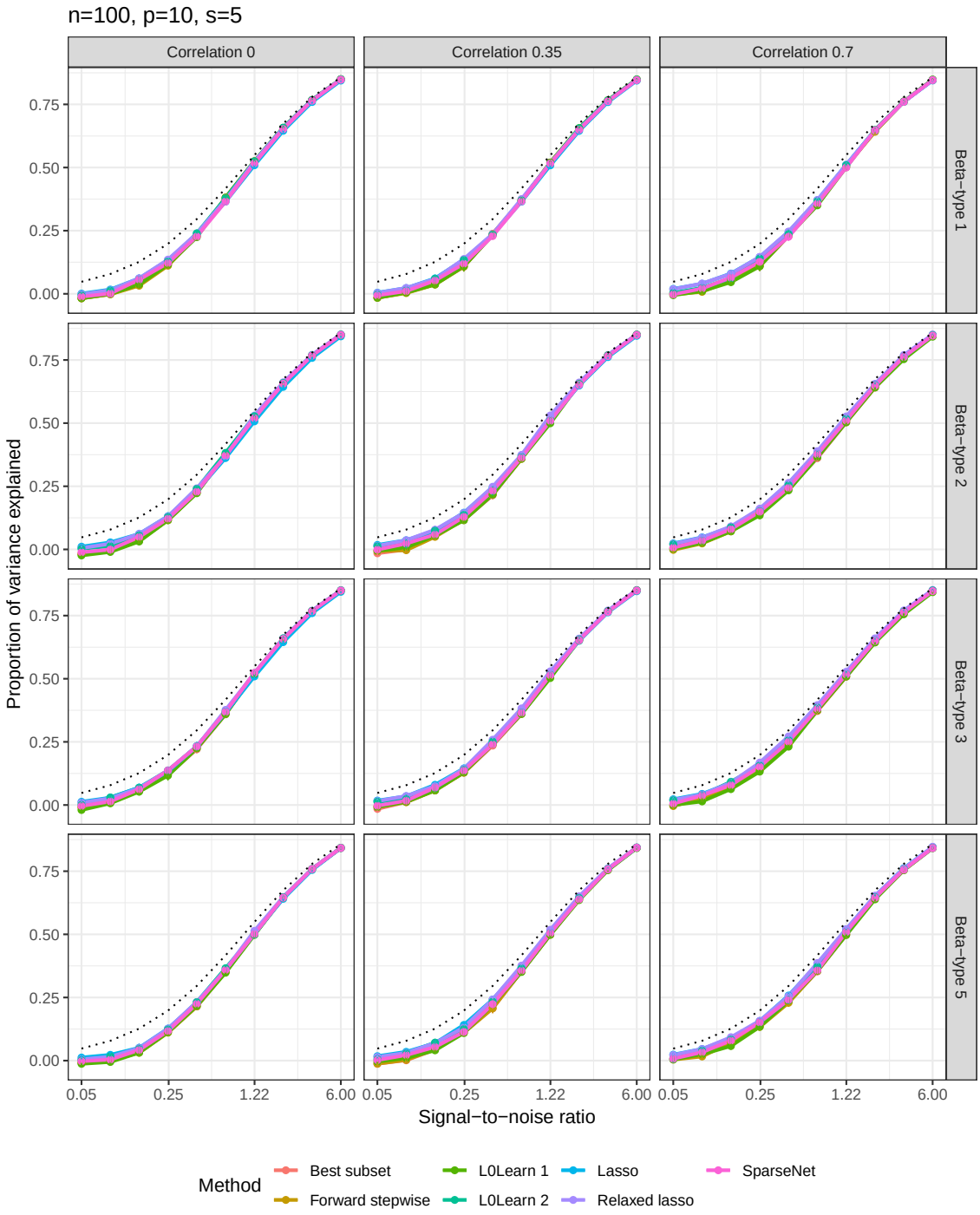
C.1.1 Relative risk (to null model)



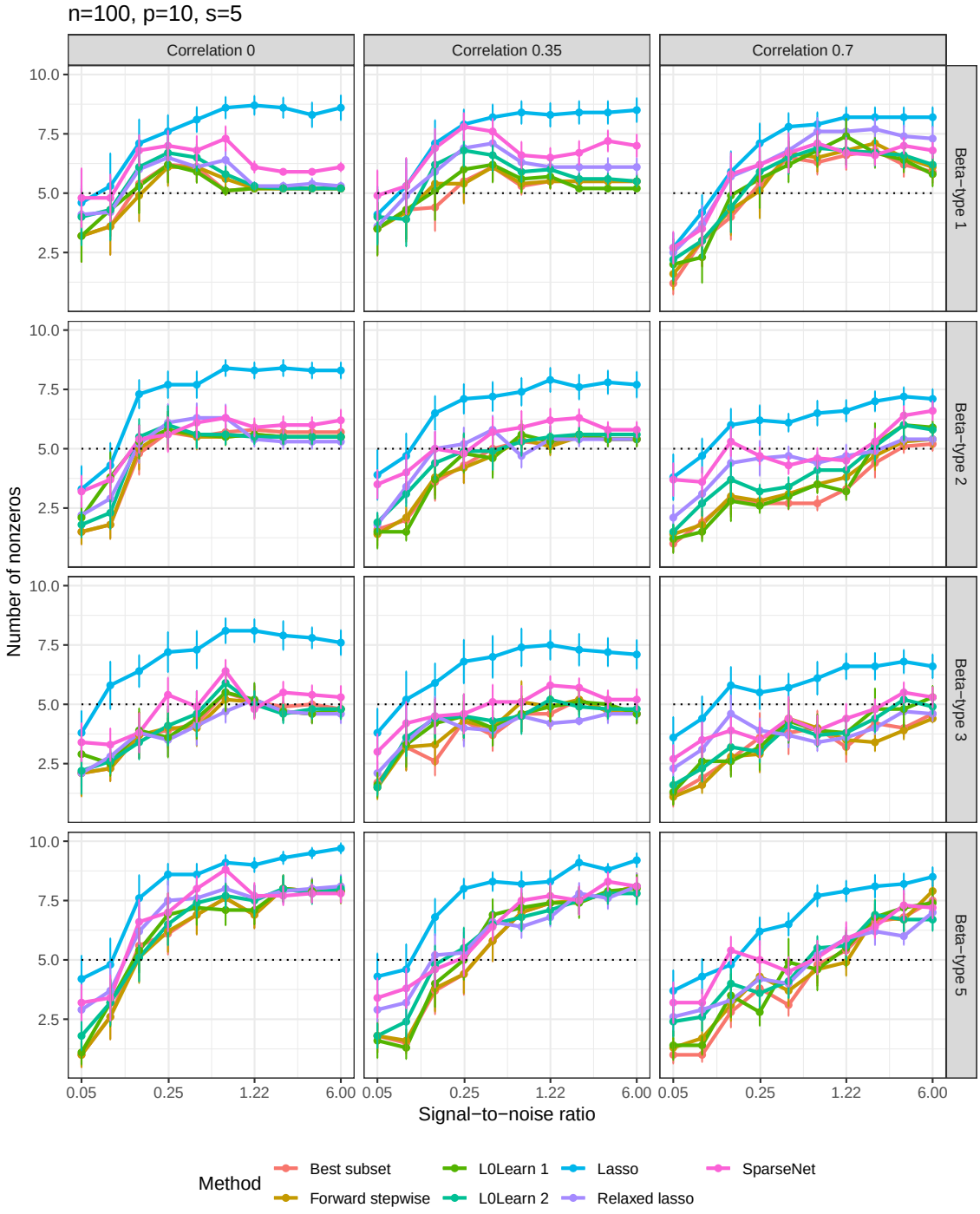
C.1.2 Relative test error (to Bayes)



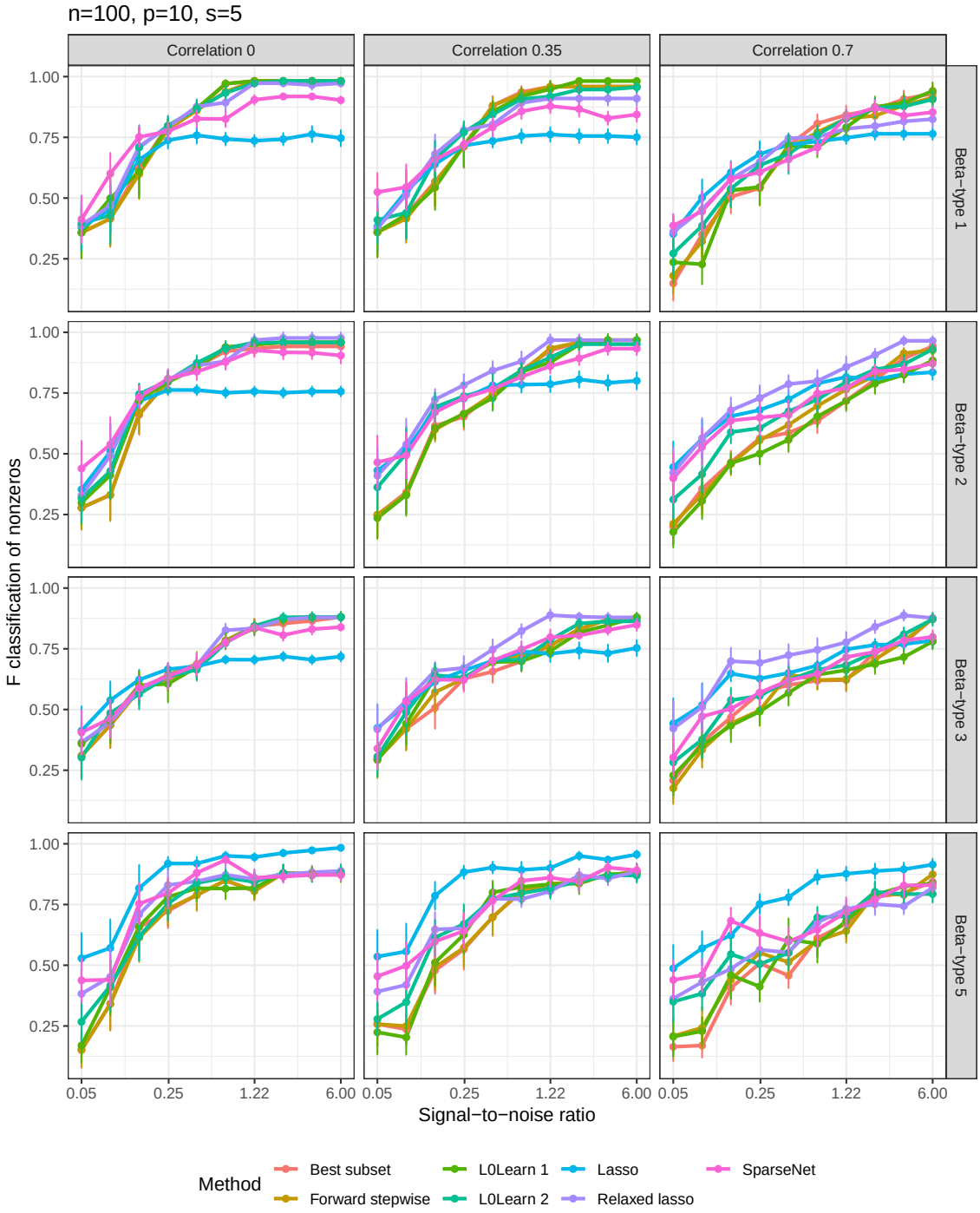
C.1.3 Proportion of variance explained



C.1.4 Number of nonzero coefficients

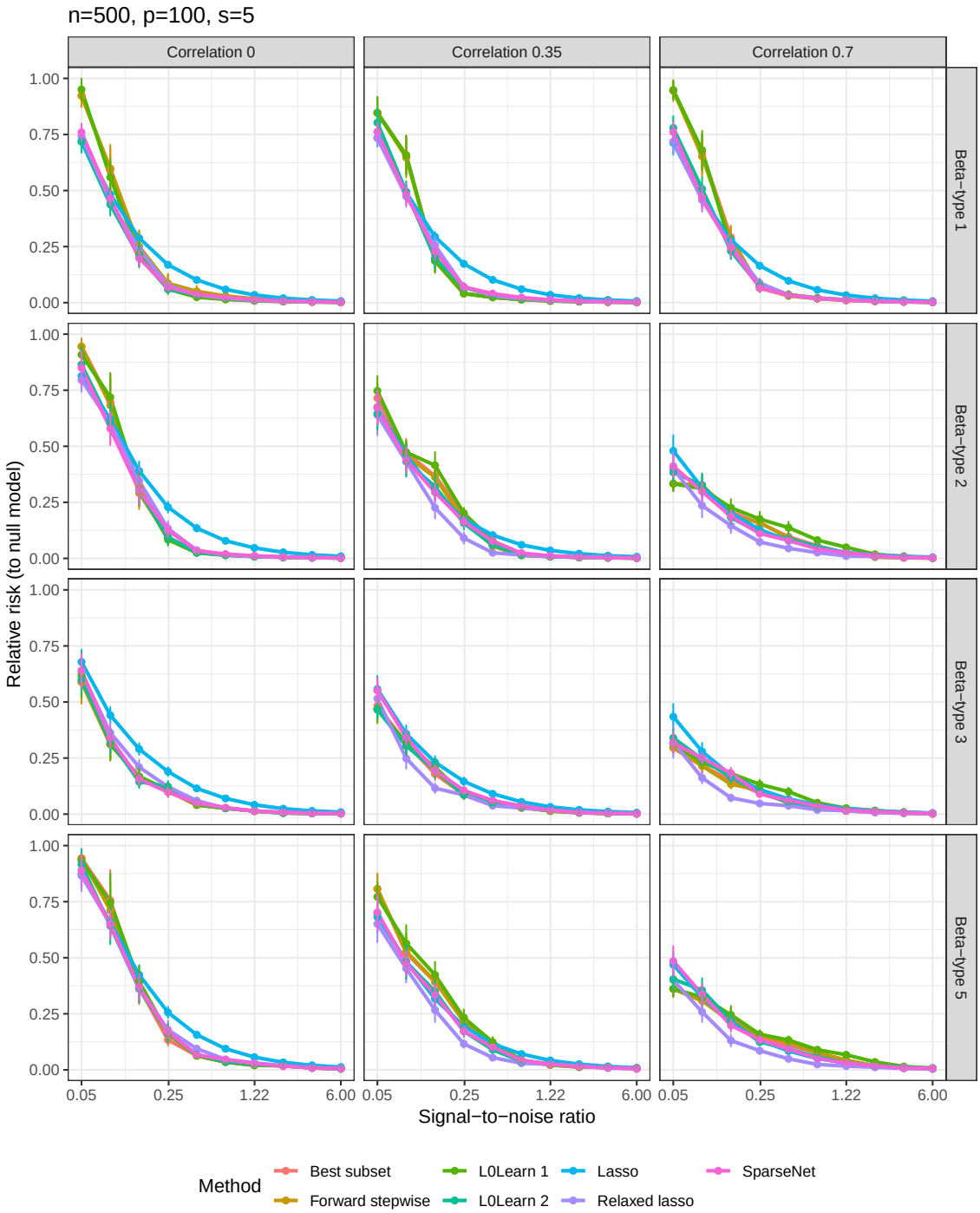


C.1.5 F-score

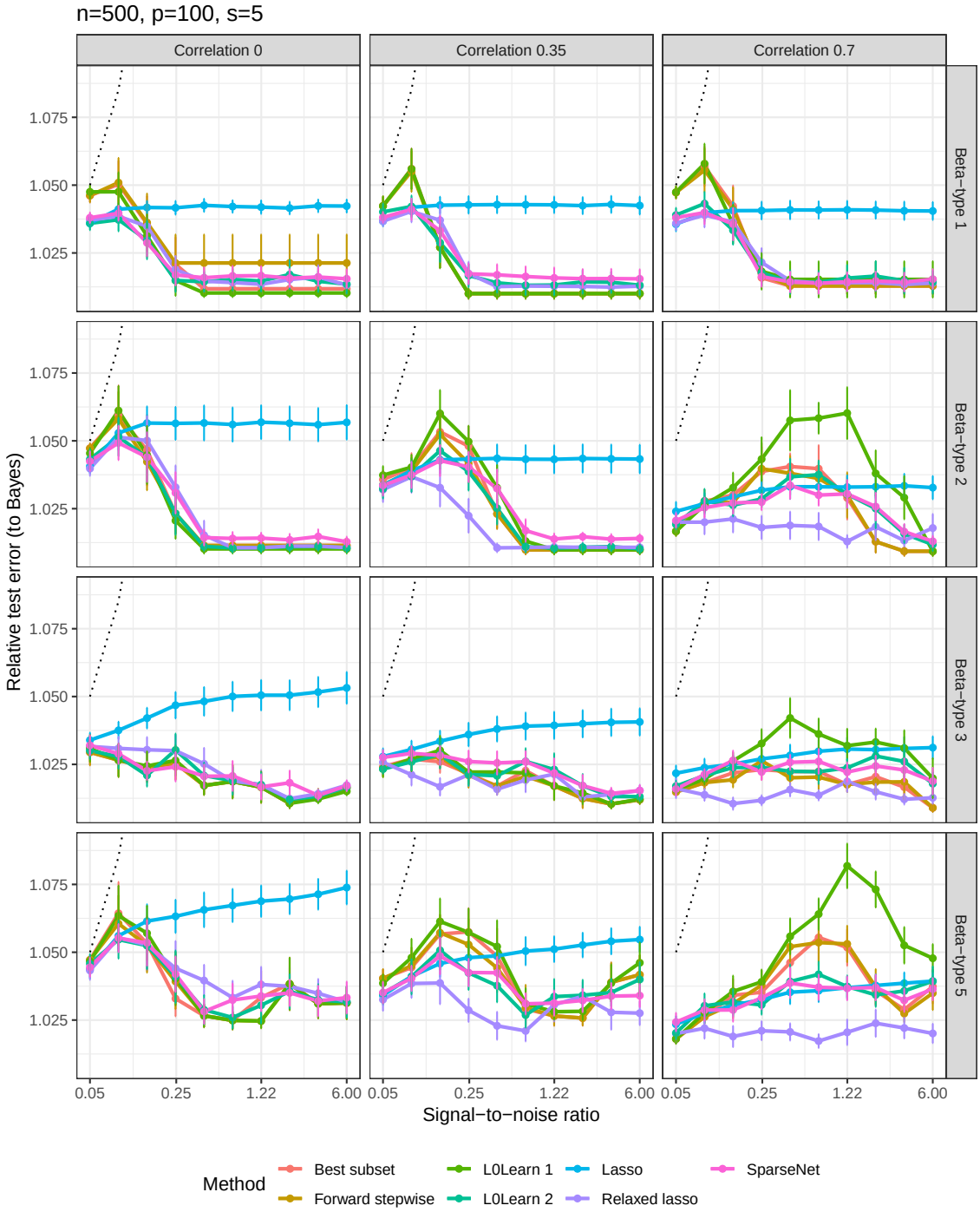

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

C.2 Medium setting: $n = 500$, $p = 100$, $s = 5$

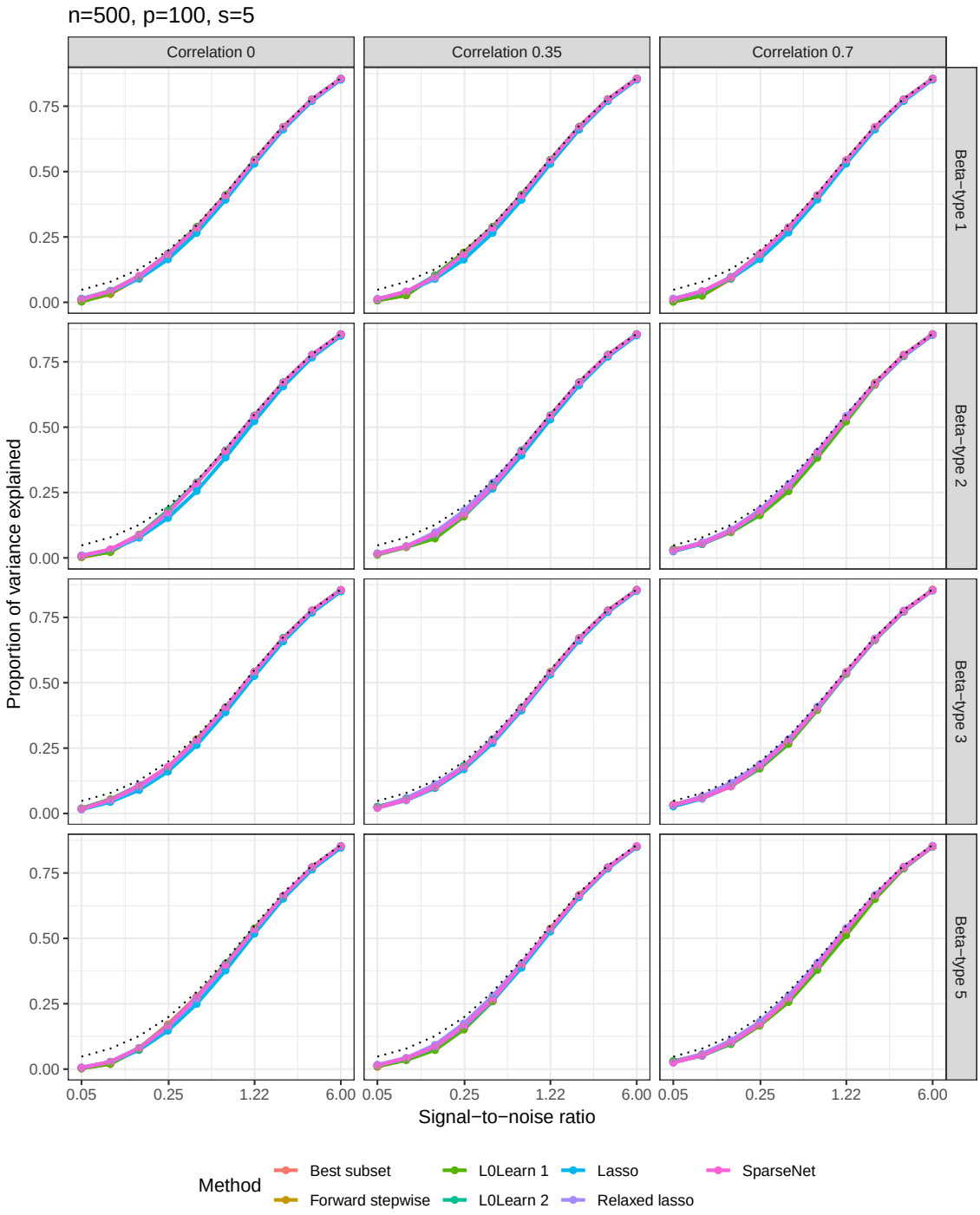
C.2.1 Relative risk (to null model)



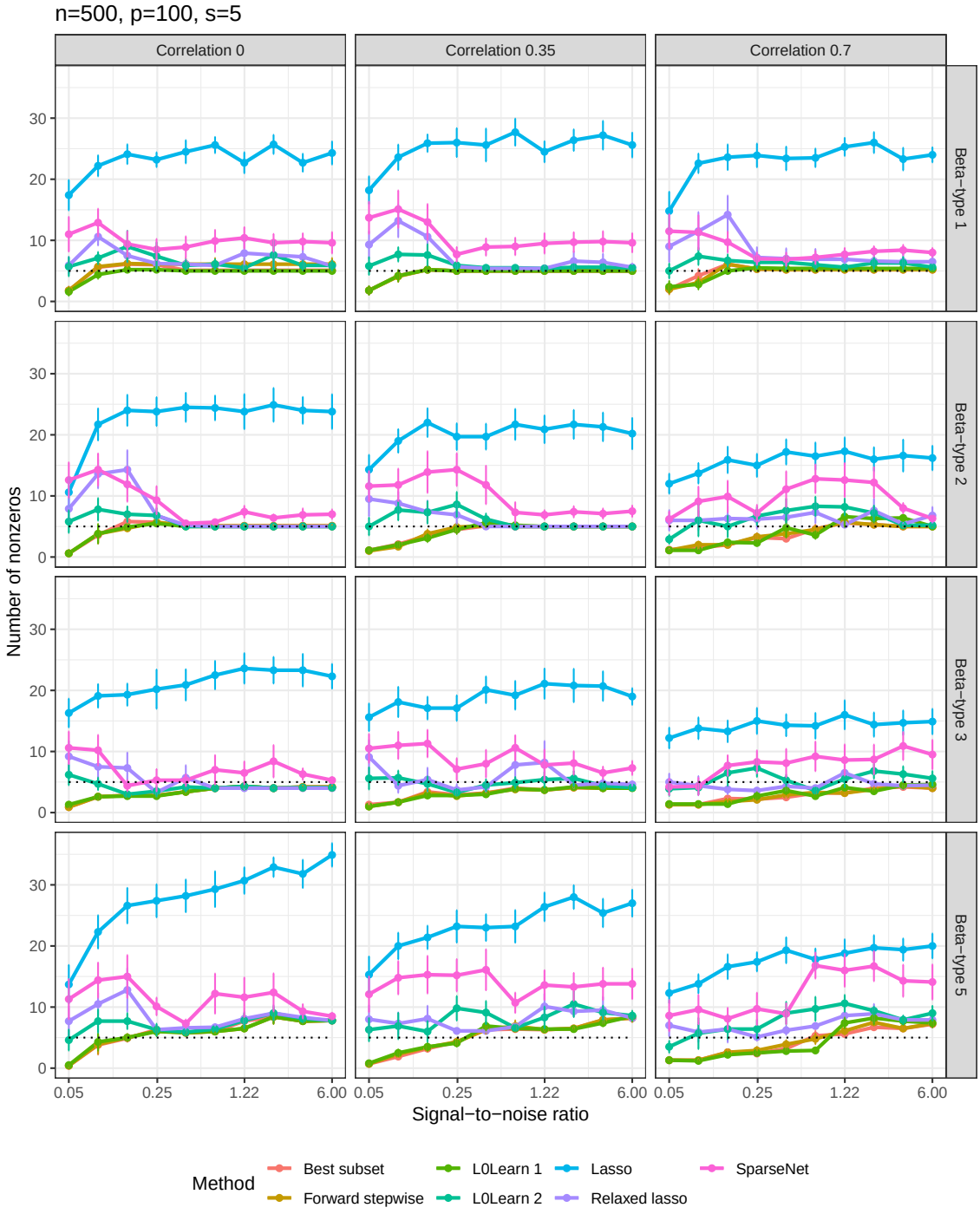
C.2.2 Relative test error (to Bayes)



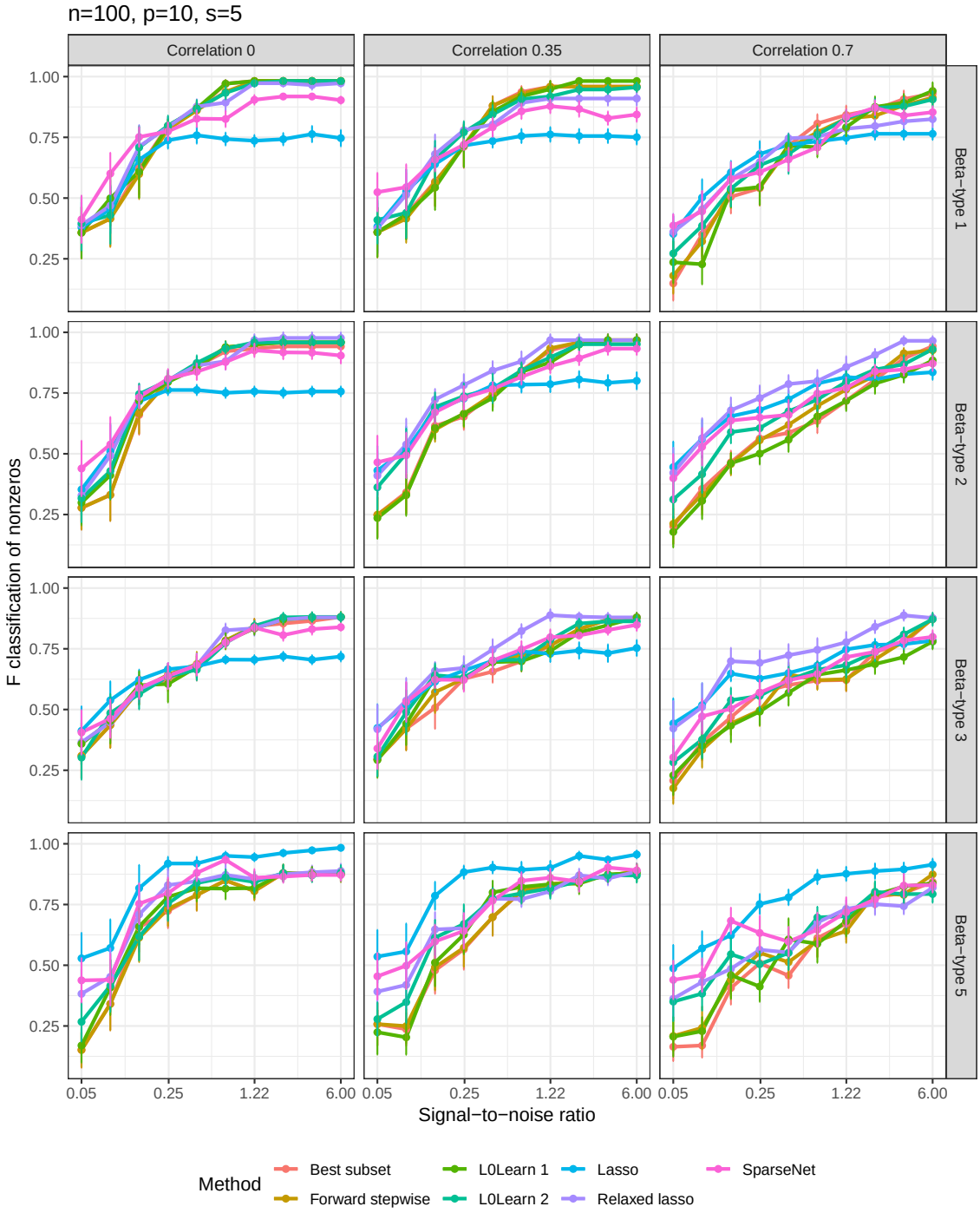
C.2.3 Proportion of variance explained



C.2.4 Number of nonzero coefficients

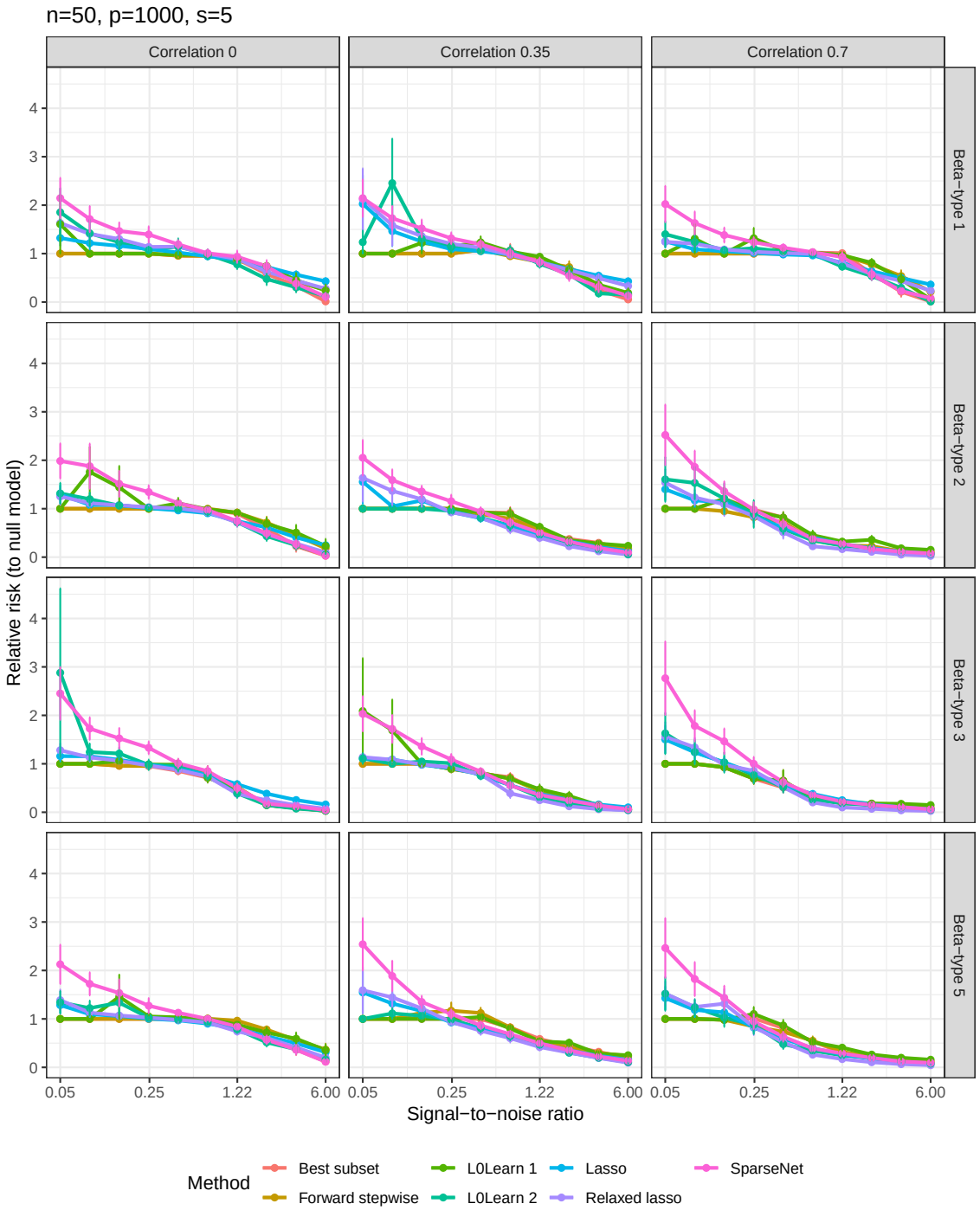


C.2.5 F-score

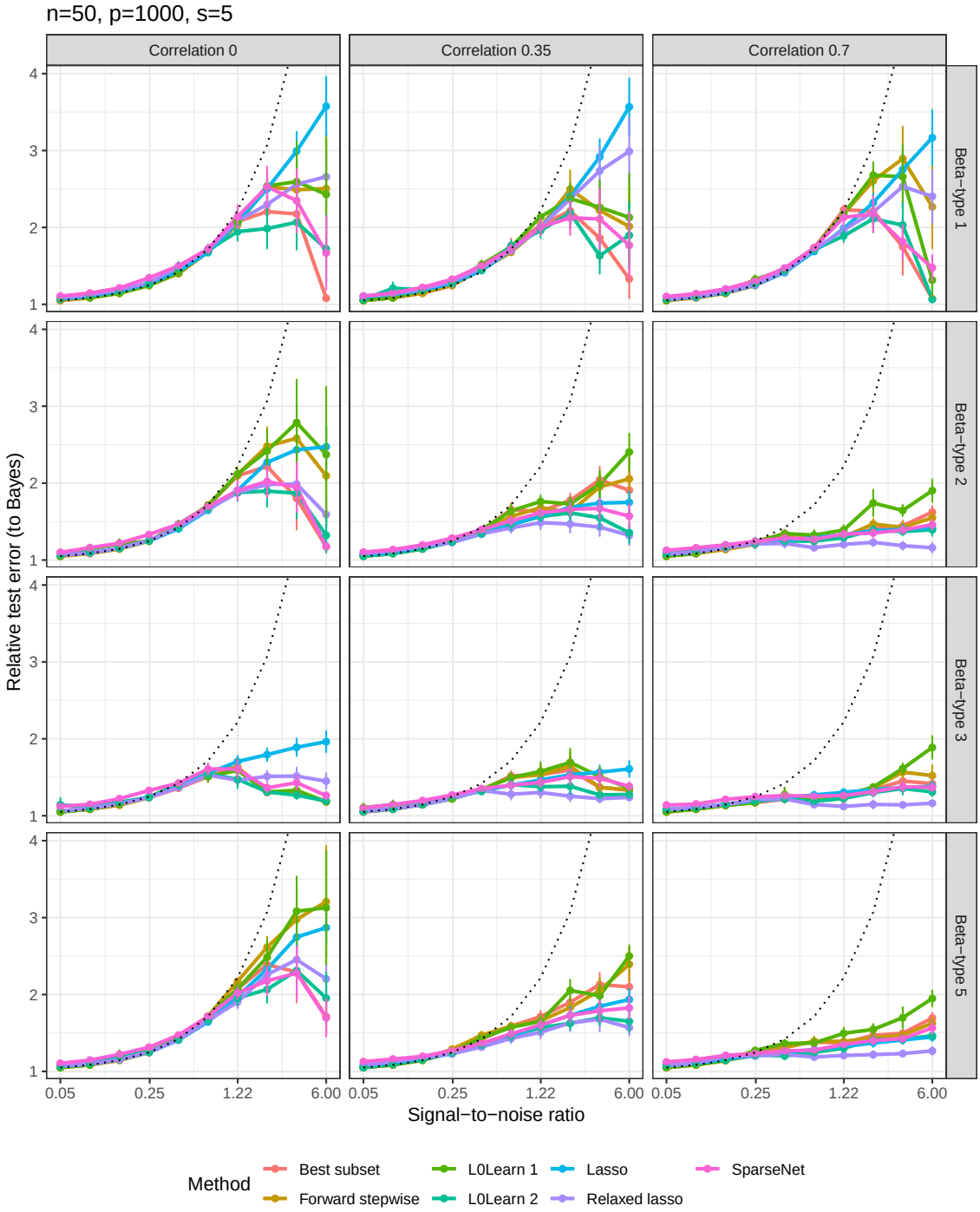

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

C.3 High-5 setting: $n = 50$, $p = 1000$, $s = 5$

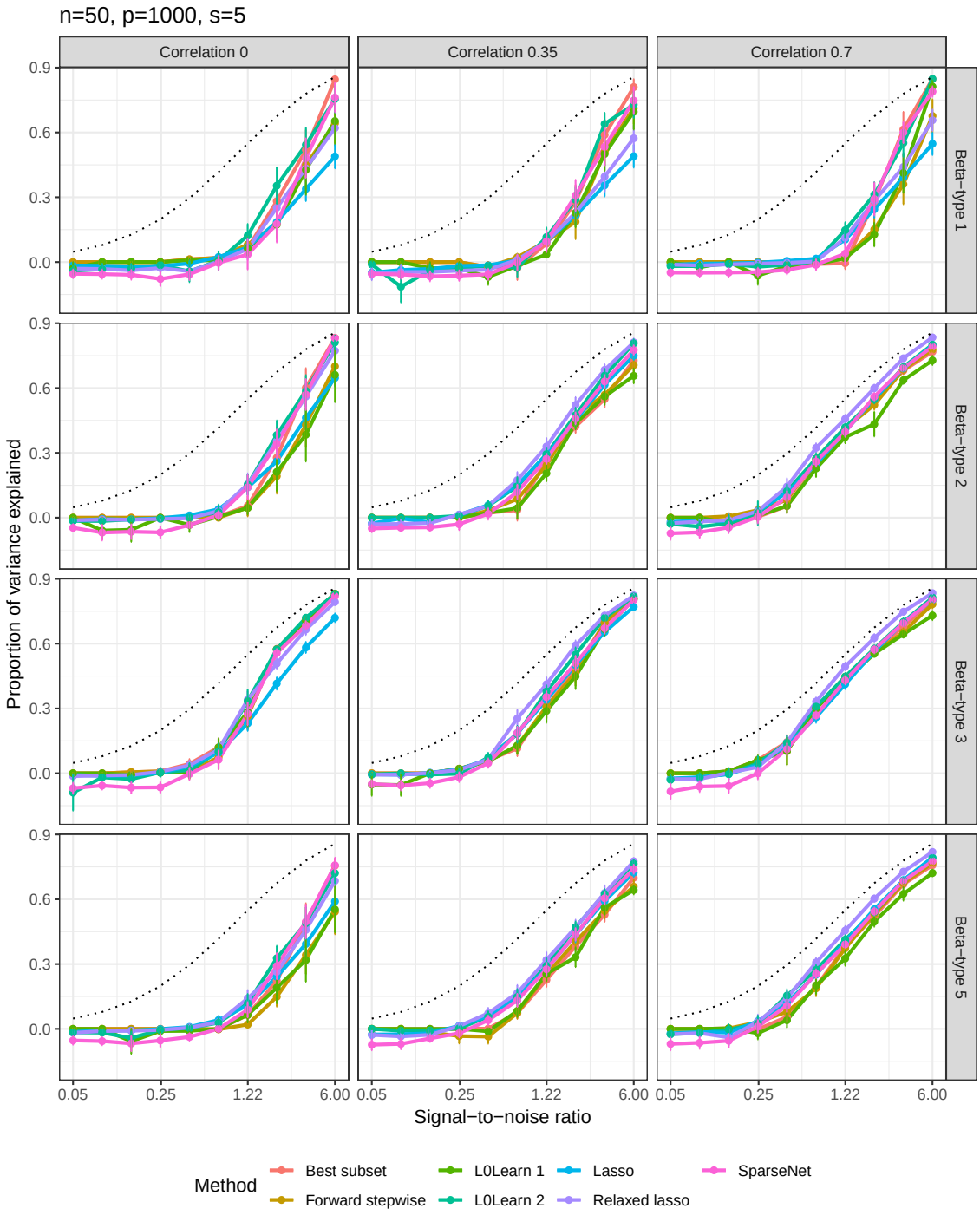
C.3.1 Relative risk (to null model)



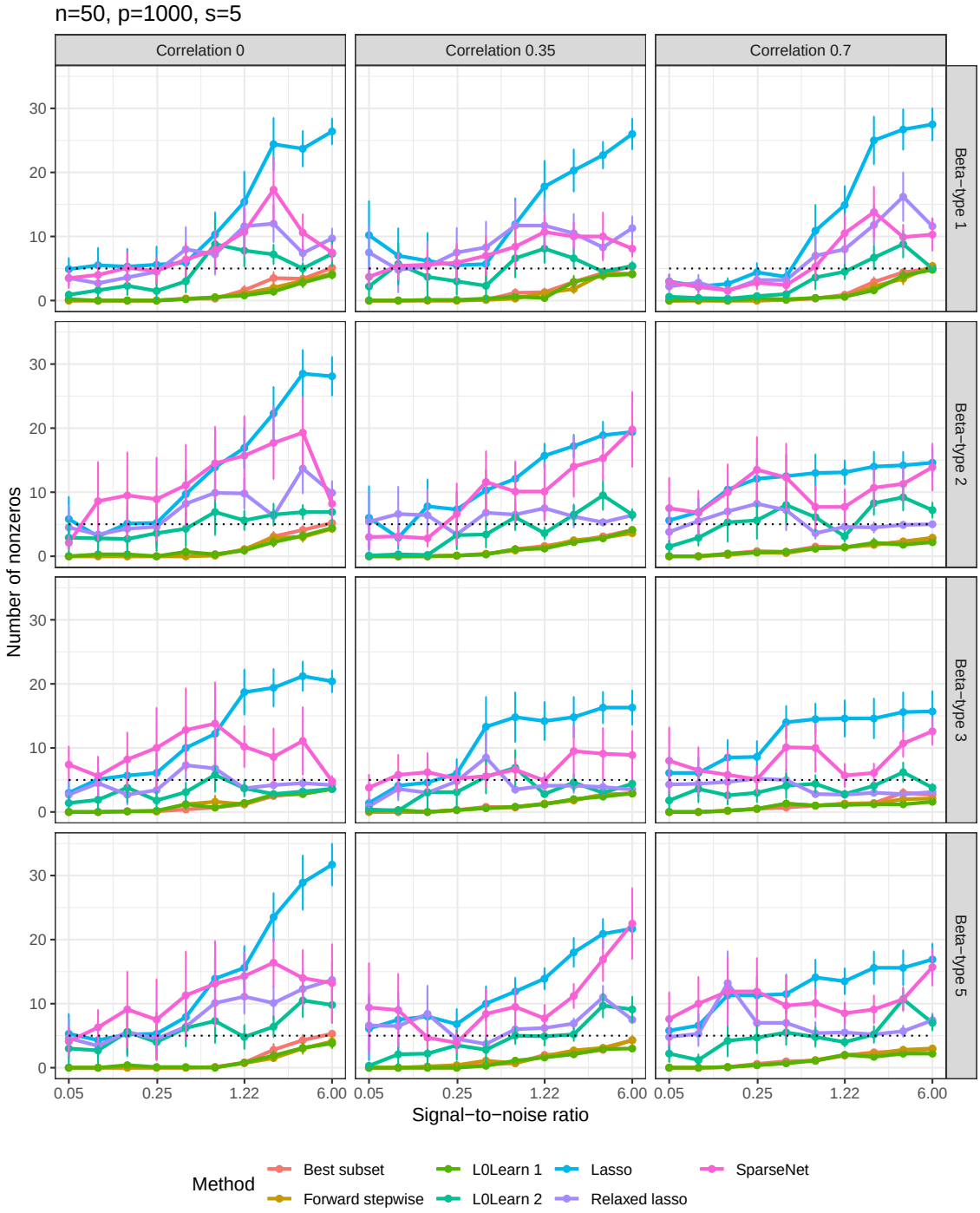
C.3.2 Relative test error (to Bayes)



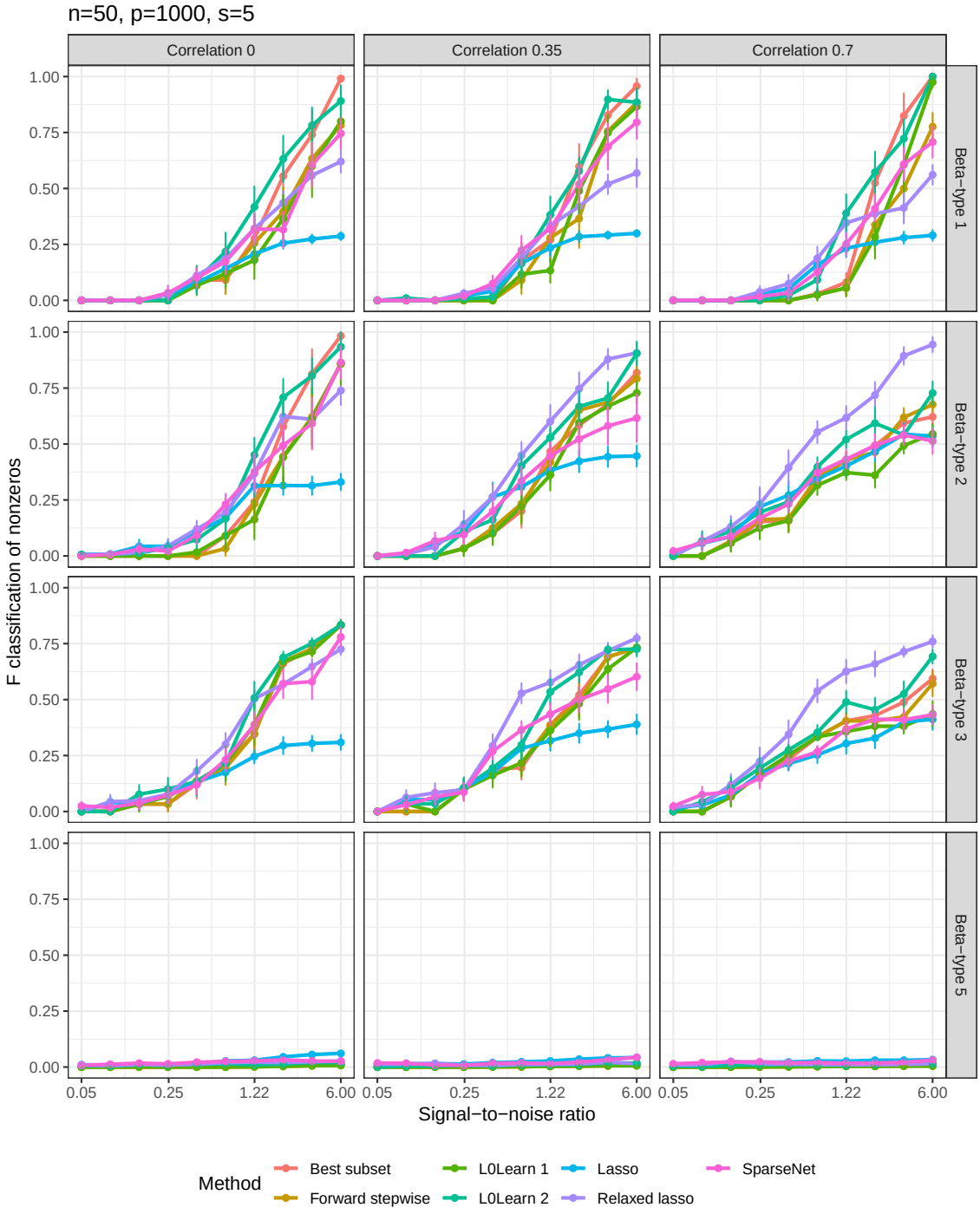
C.3.3 Proportion of variance explained



C.3.4 Number of nonzero coefficients

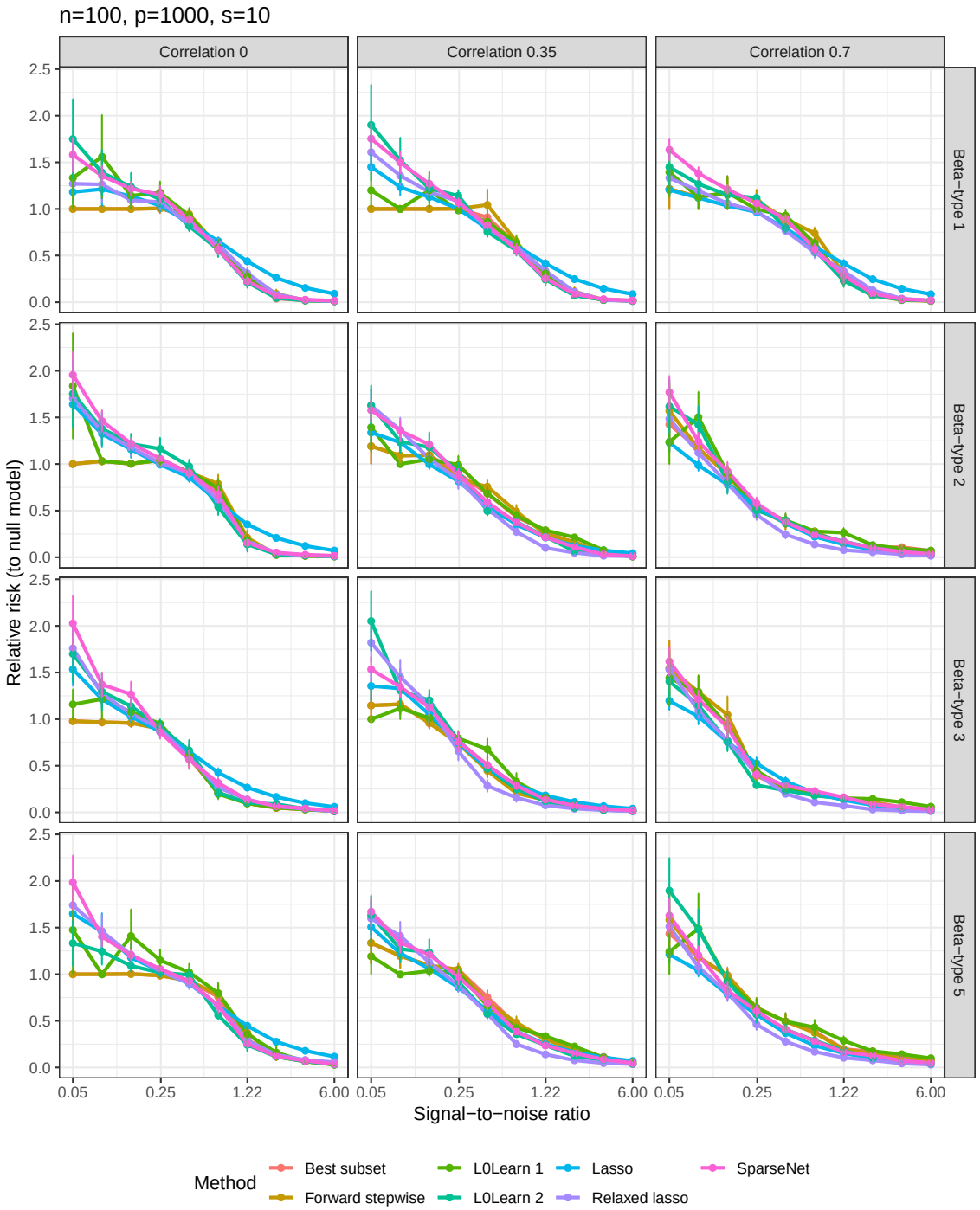


C.3.5 F-score

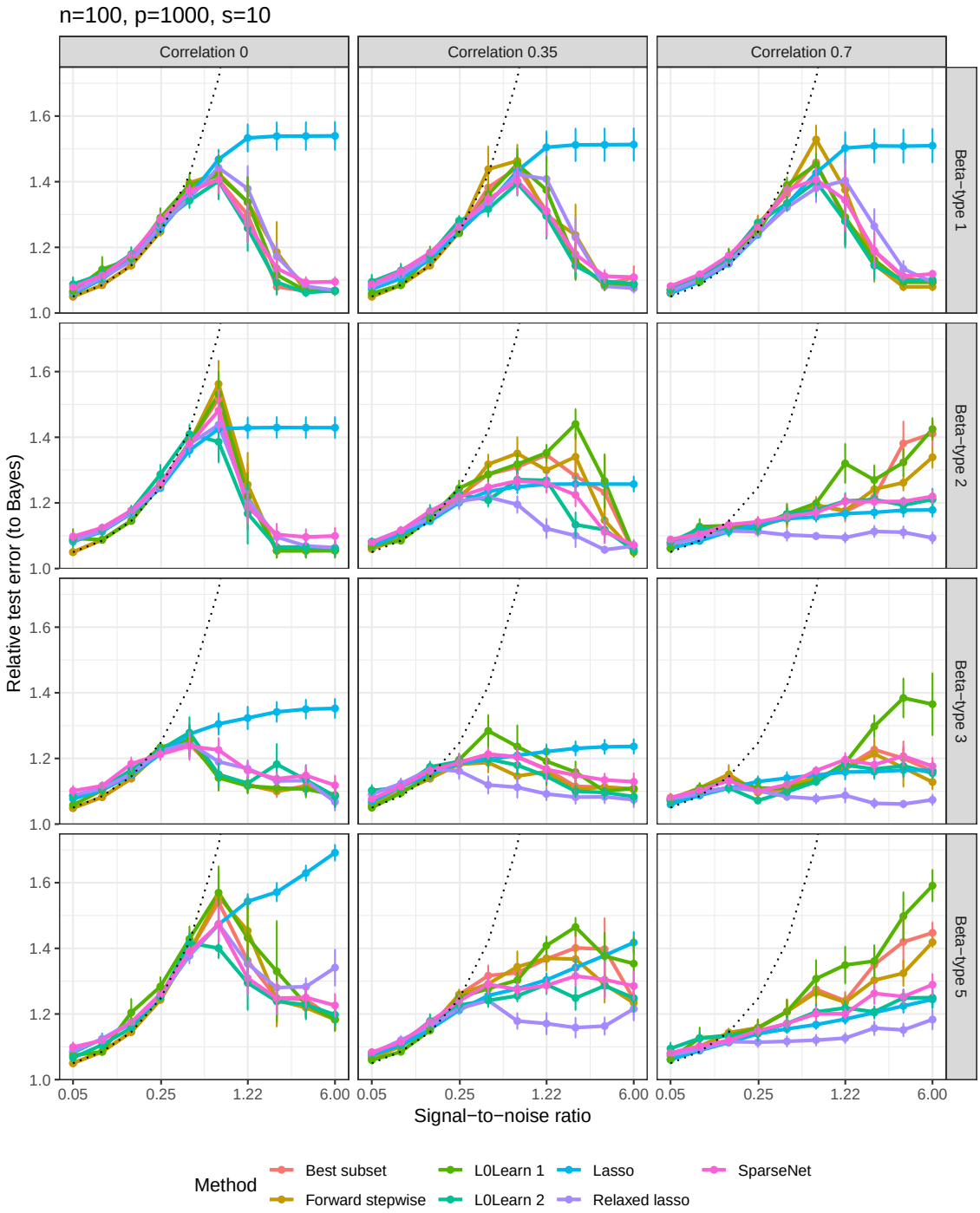

n=50, p=1000, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

C.4 High-10 setting: $n = 100$, $p = 1000$, $s = 10$

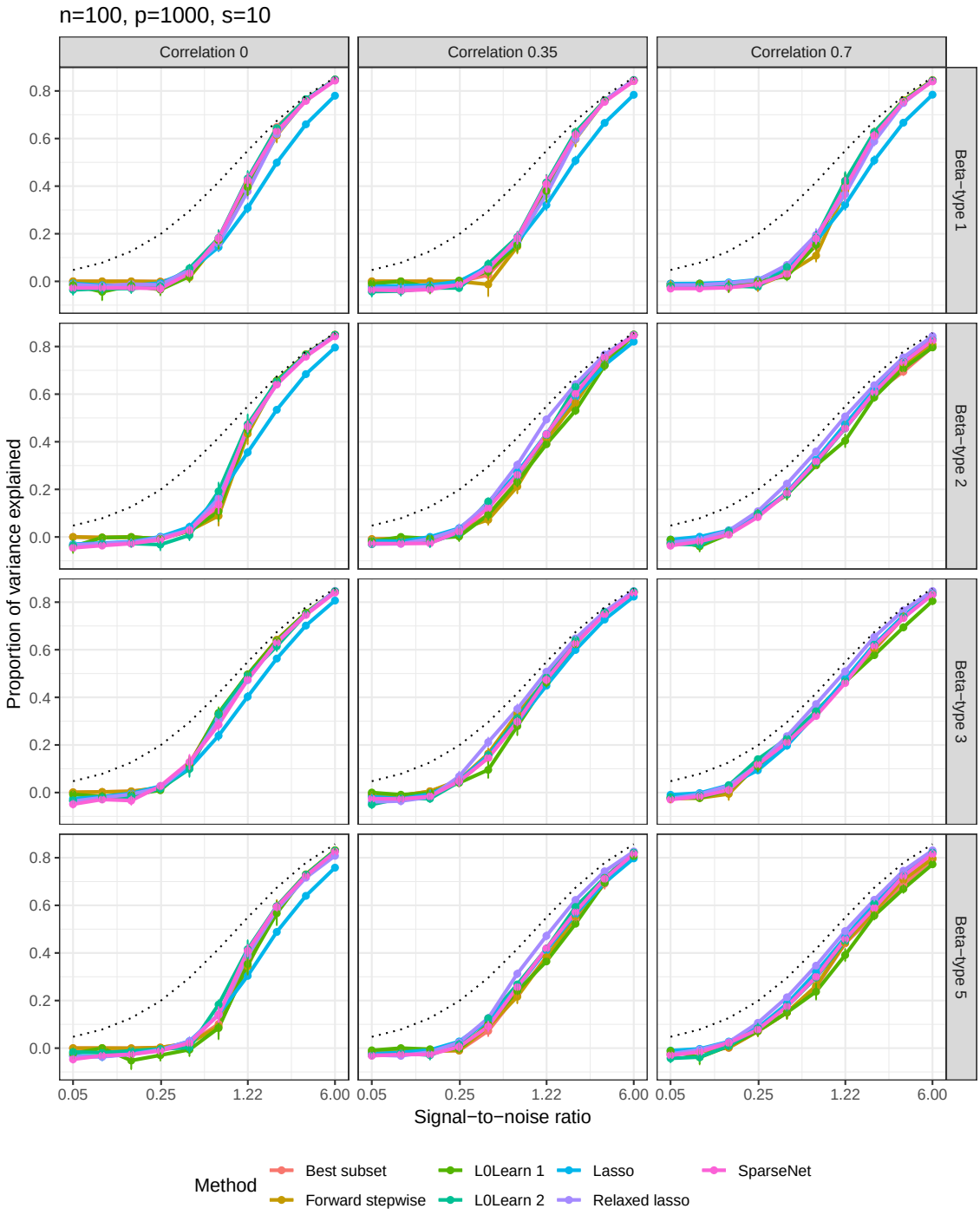
C.4.1 Relative risk (to null model)



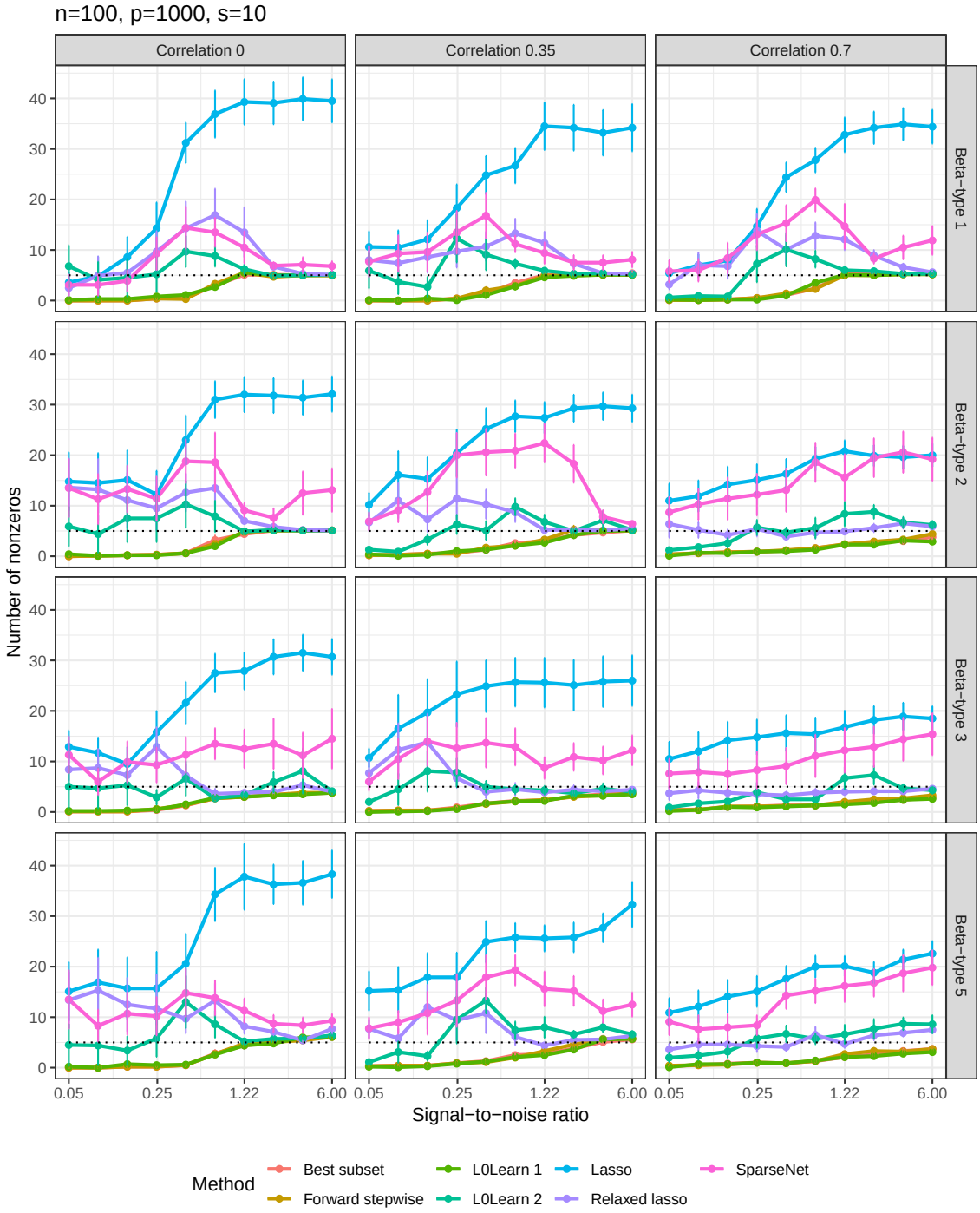
C.4.2 Relative test error (to Bayes)



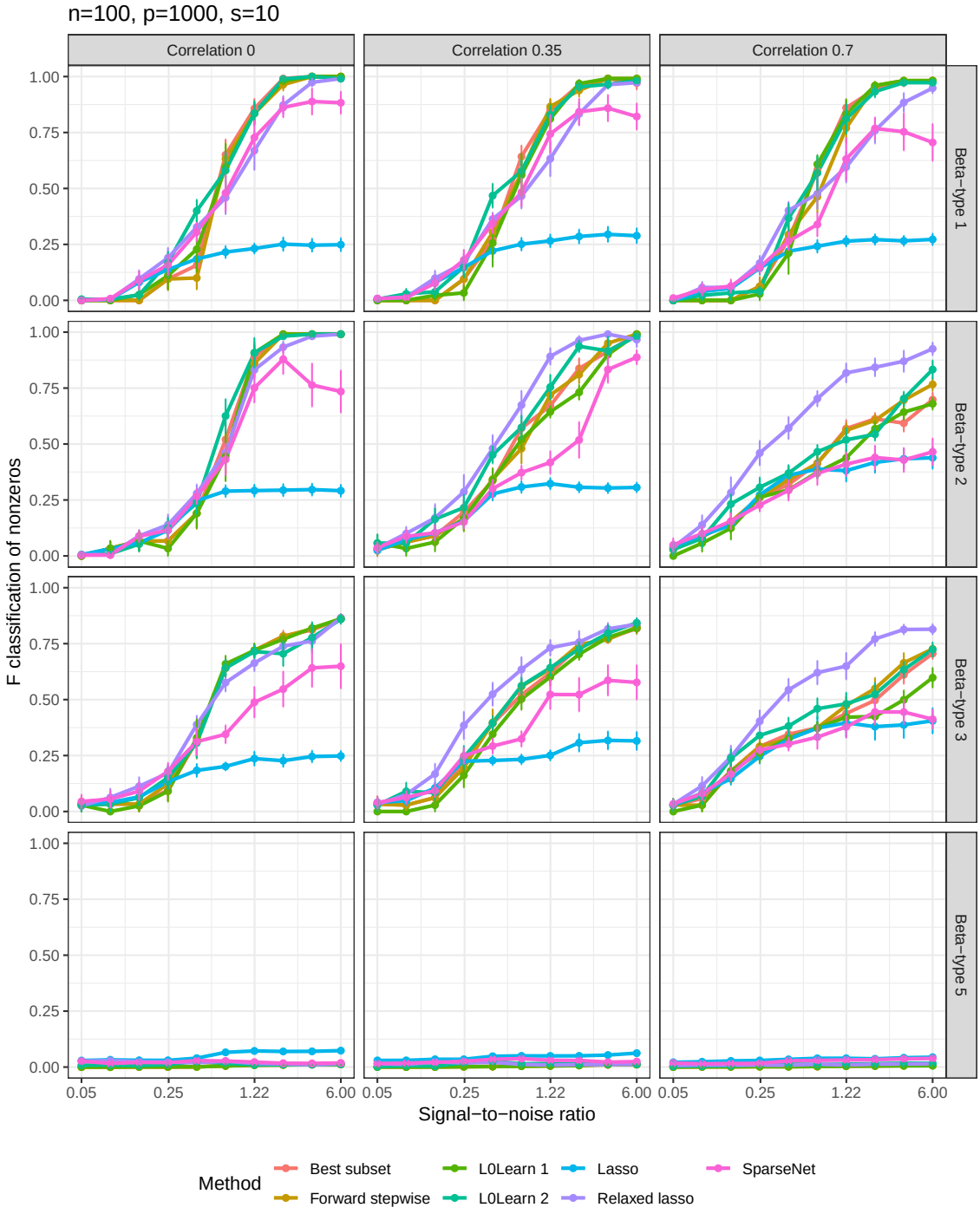
C.4.3 Proportion of variance explained



C.4.4 Number of nonzero coefficients



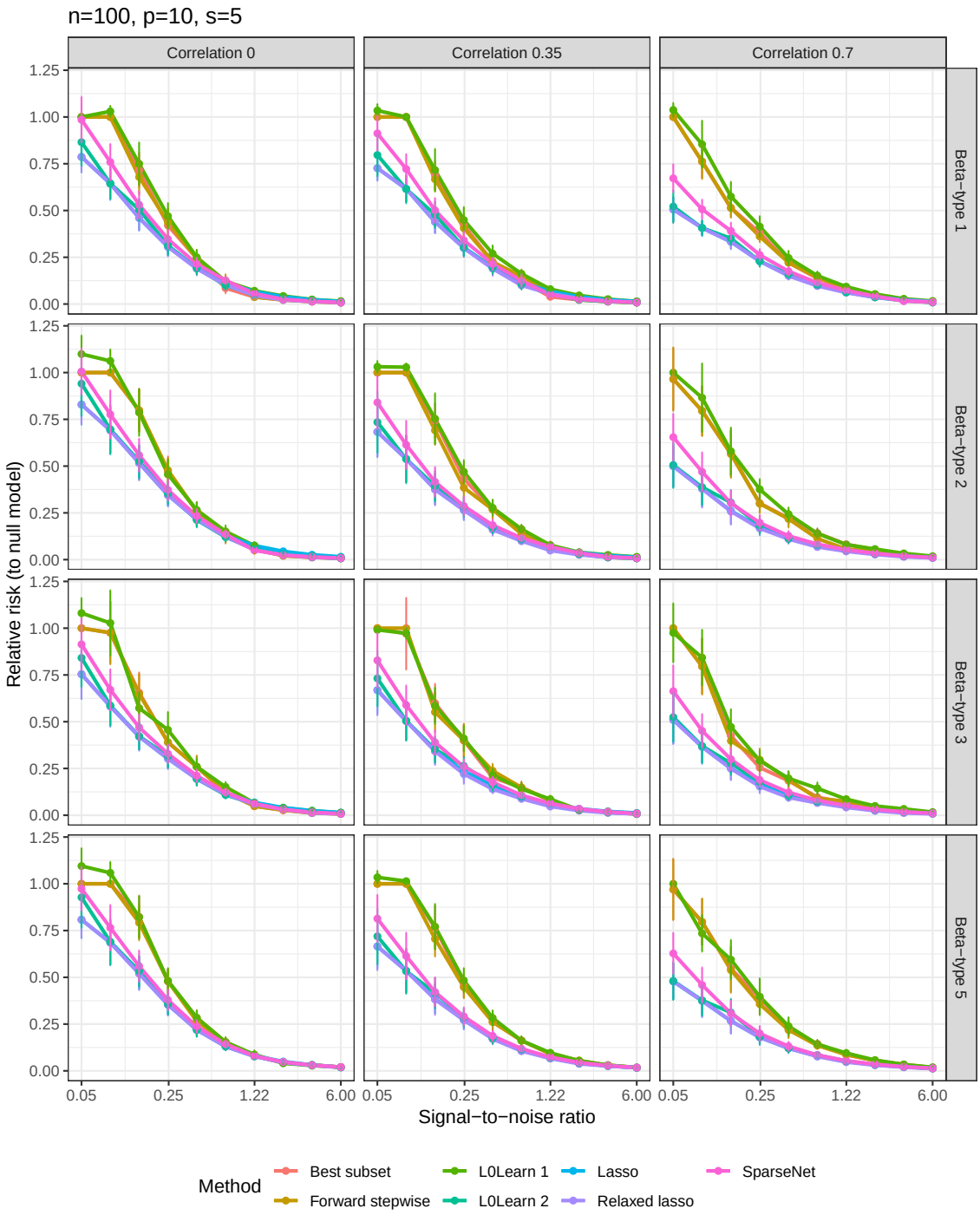
C.4.5 F-score


n=100, p=1000, s=10
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

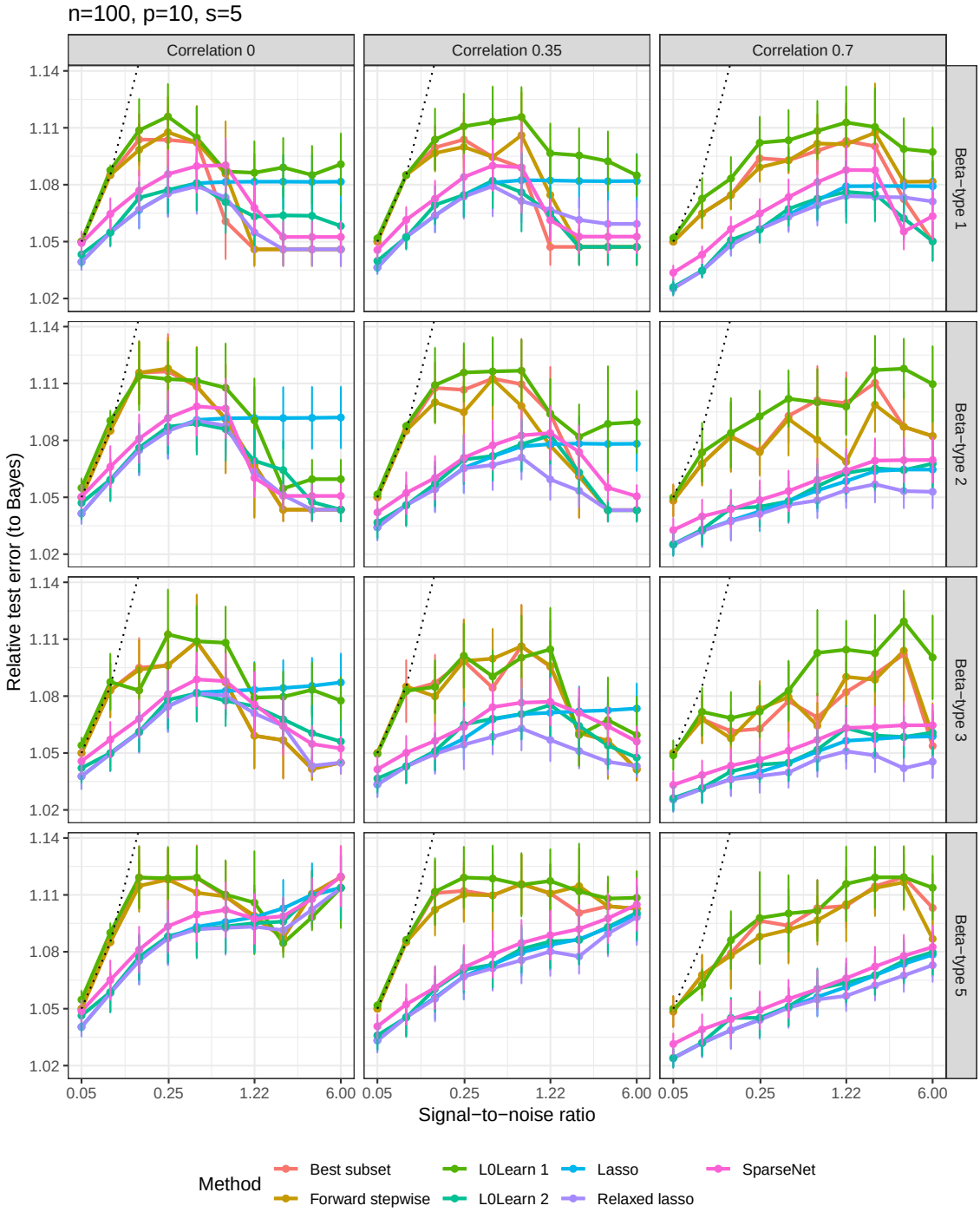
D Oracle tuning: more methods

D.1 Low setting: $n = 100$, $p = 10$, $s = 5$

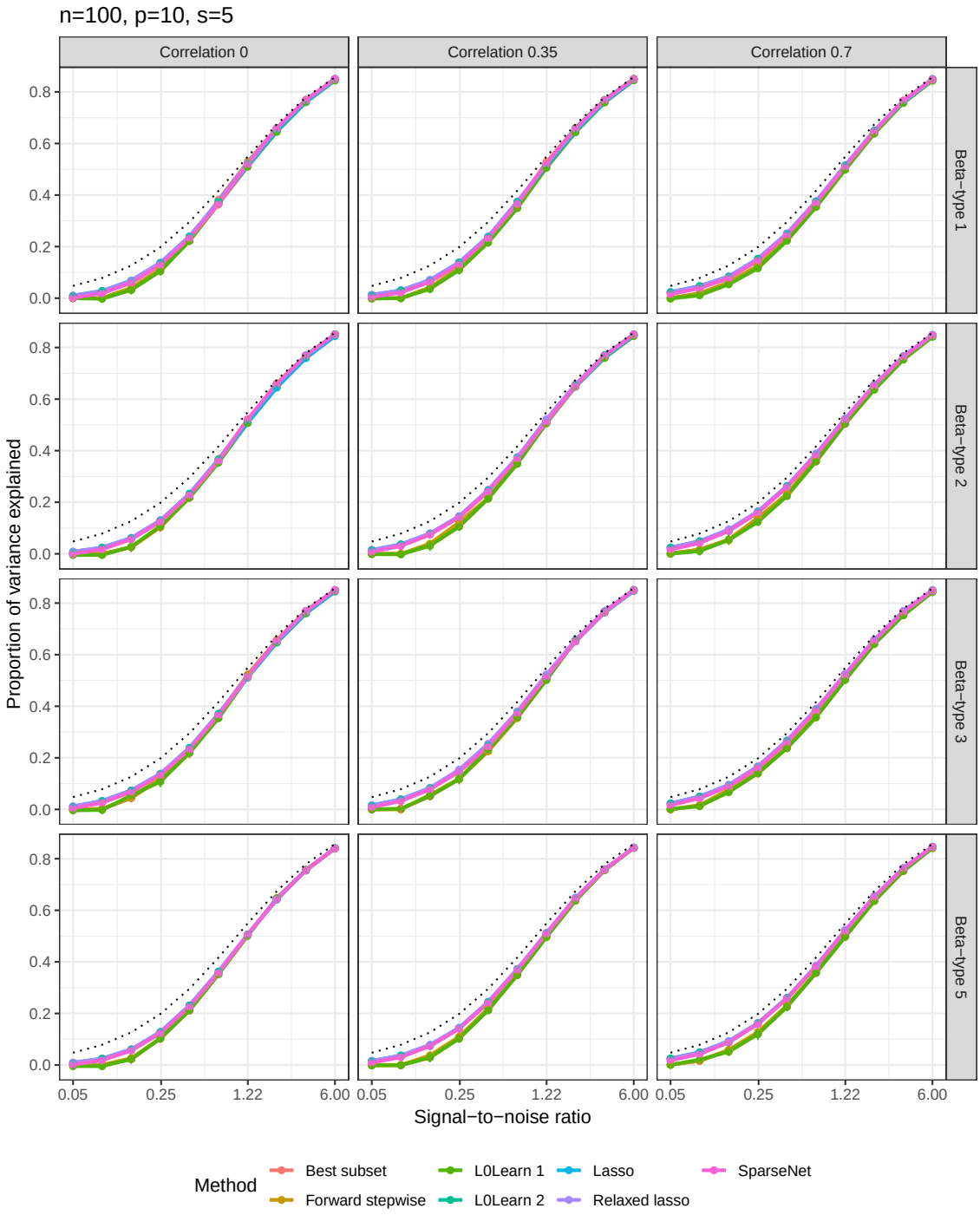
D.1.1 Relative risk (to null model)



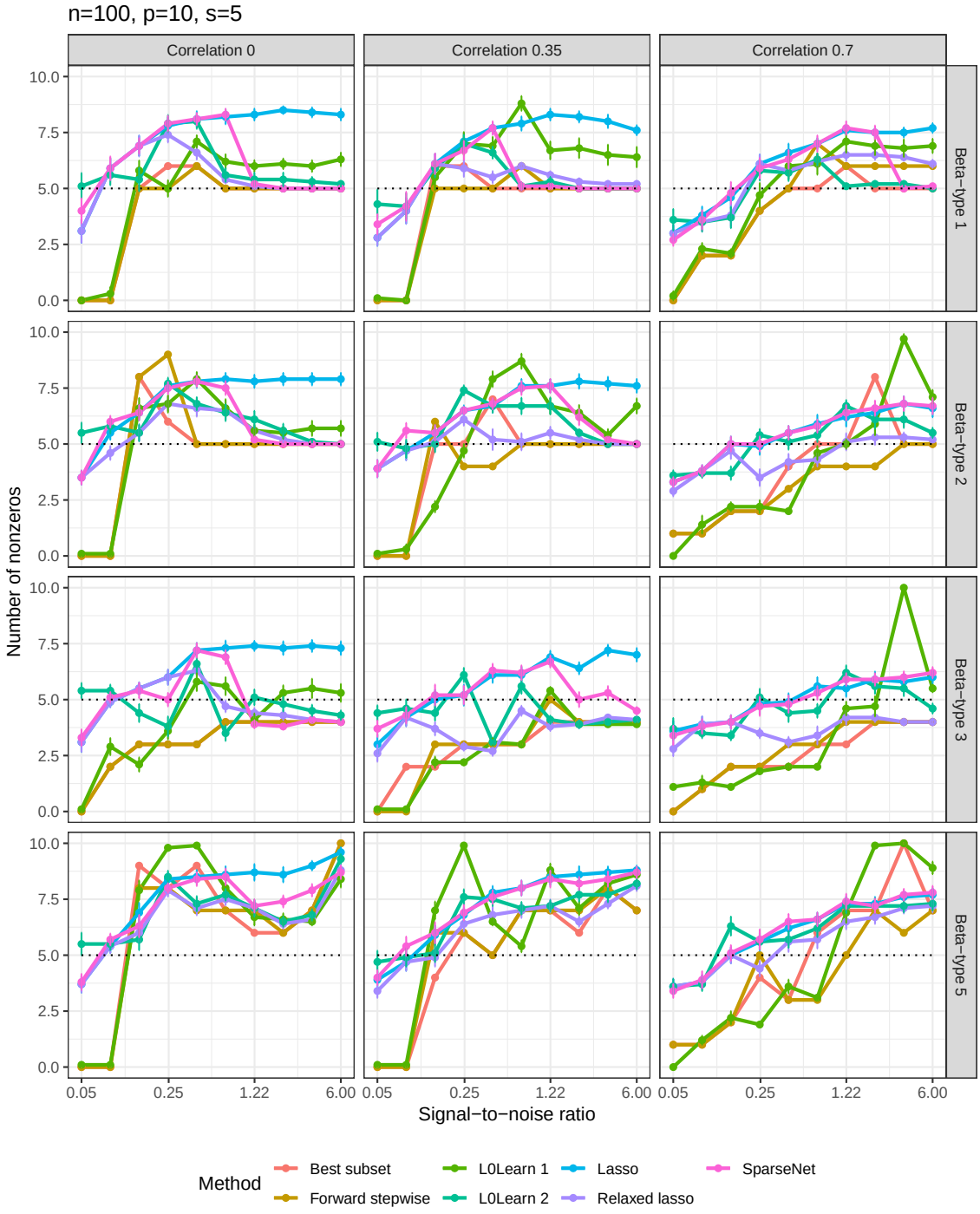
D.1.2 Relative test error (to Bayes)



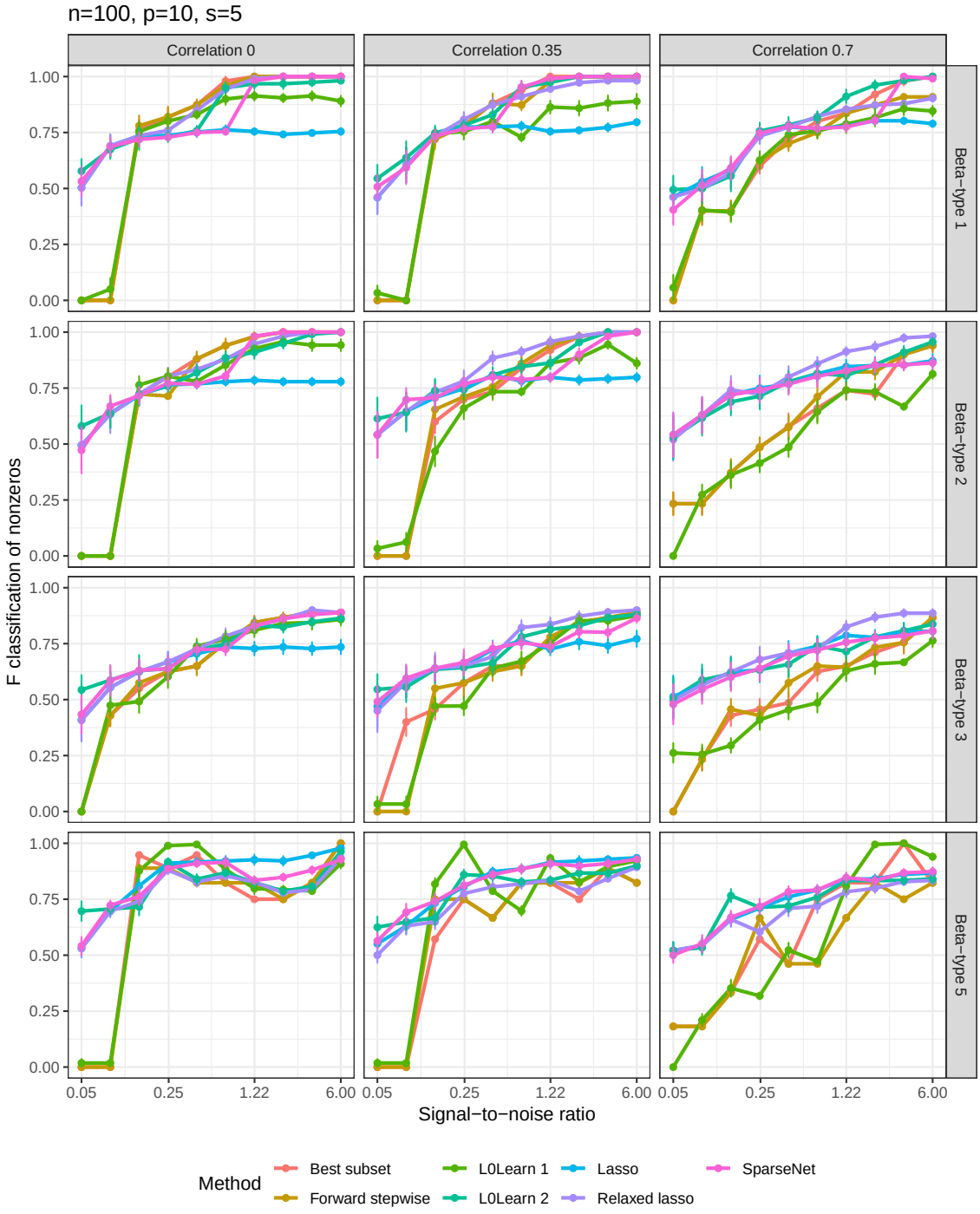
D.1.3 Proportion of variance explained



D.1.4 Number of nonzero coefficients

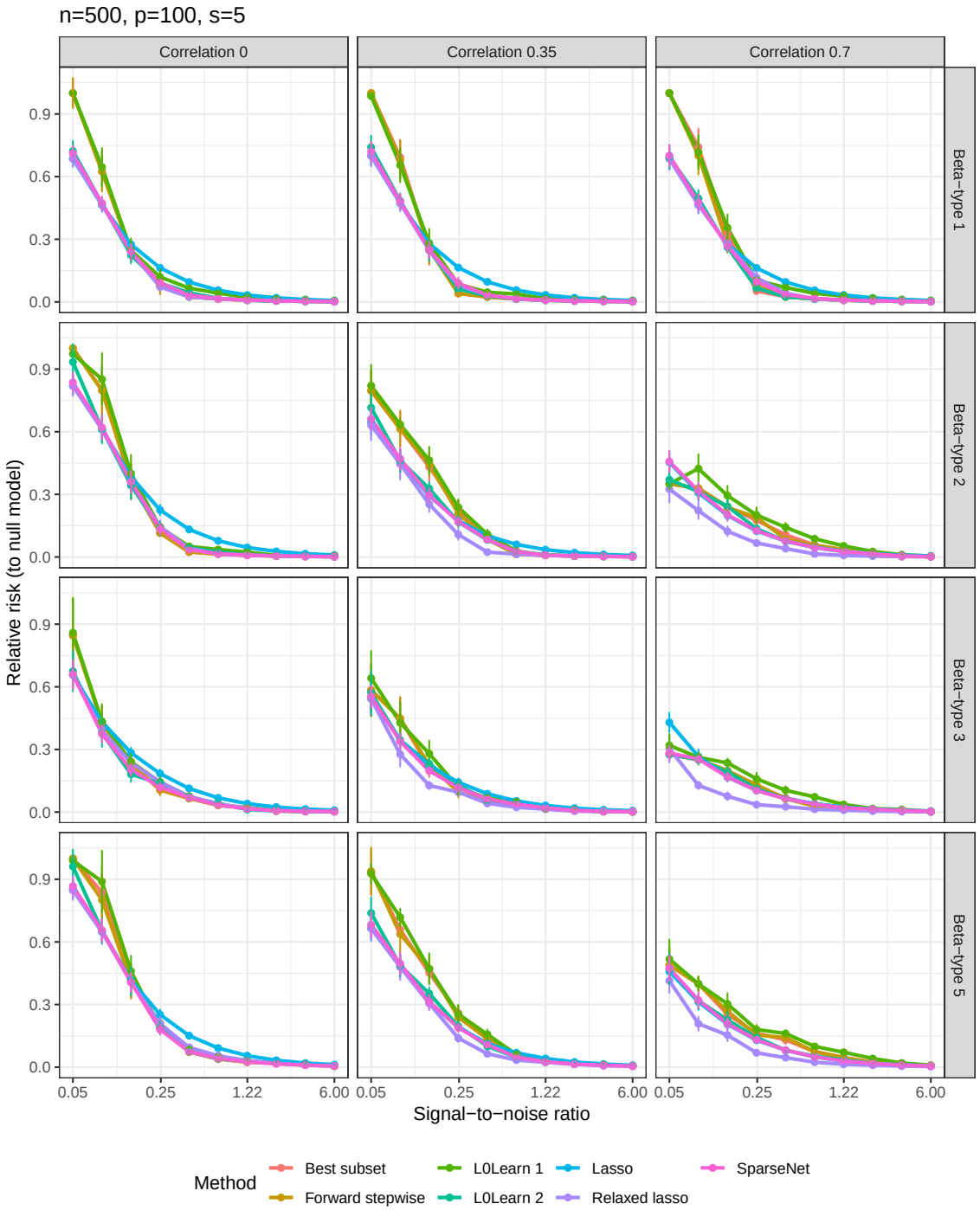


D.1.5 F-score

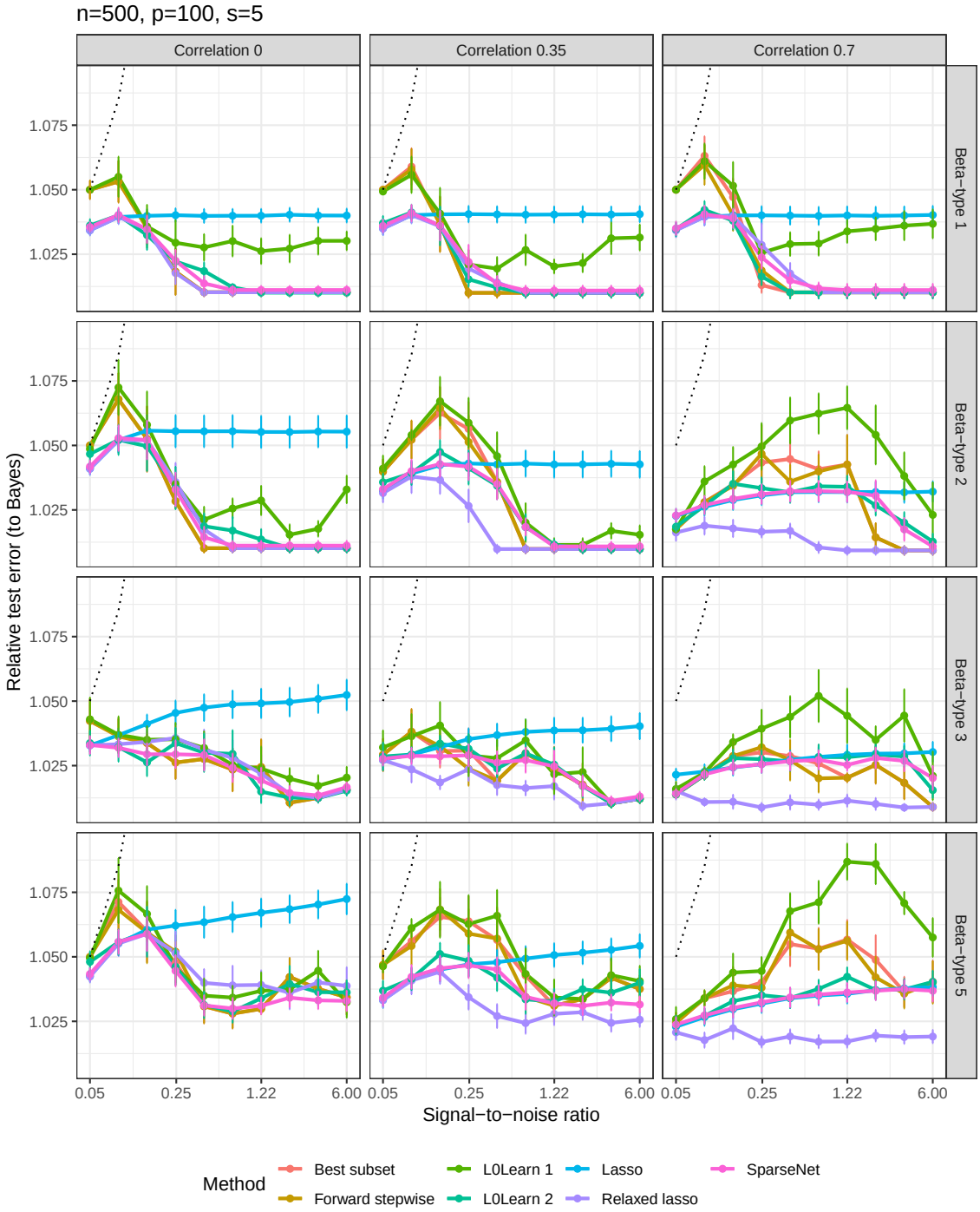

n=100, p=10, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

D.2 Medium setting: $n = 500$, $p = 100$, $s = 5$

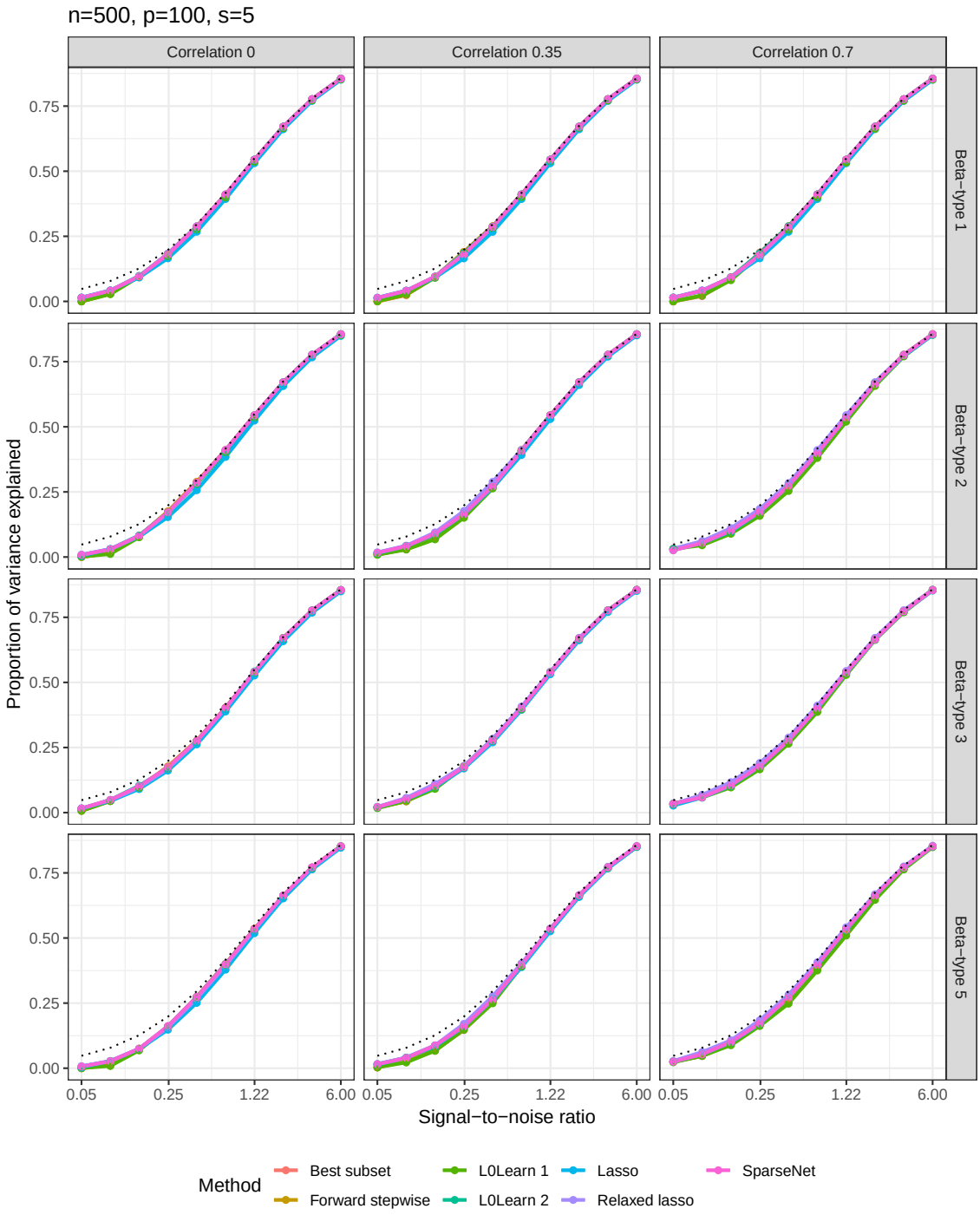
D.2.1 Relative risk (to null model)



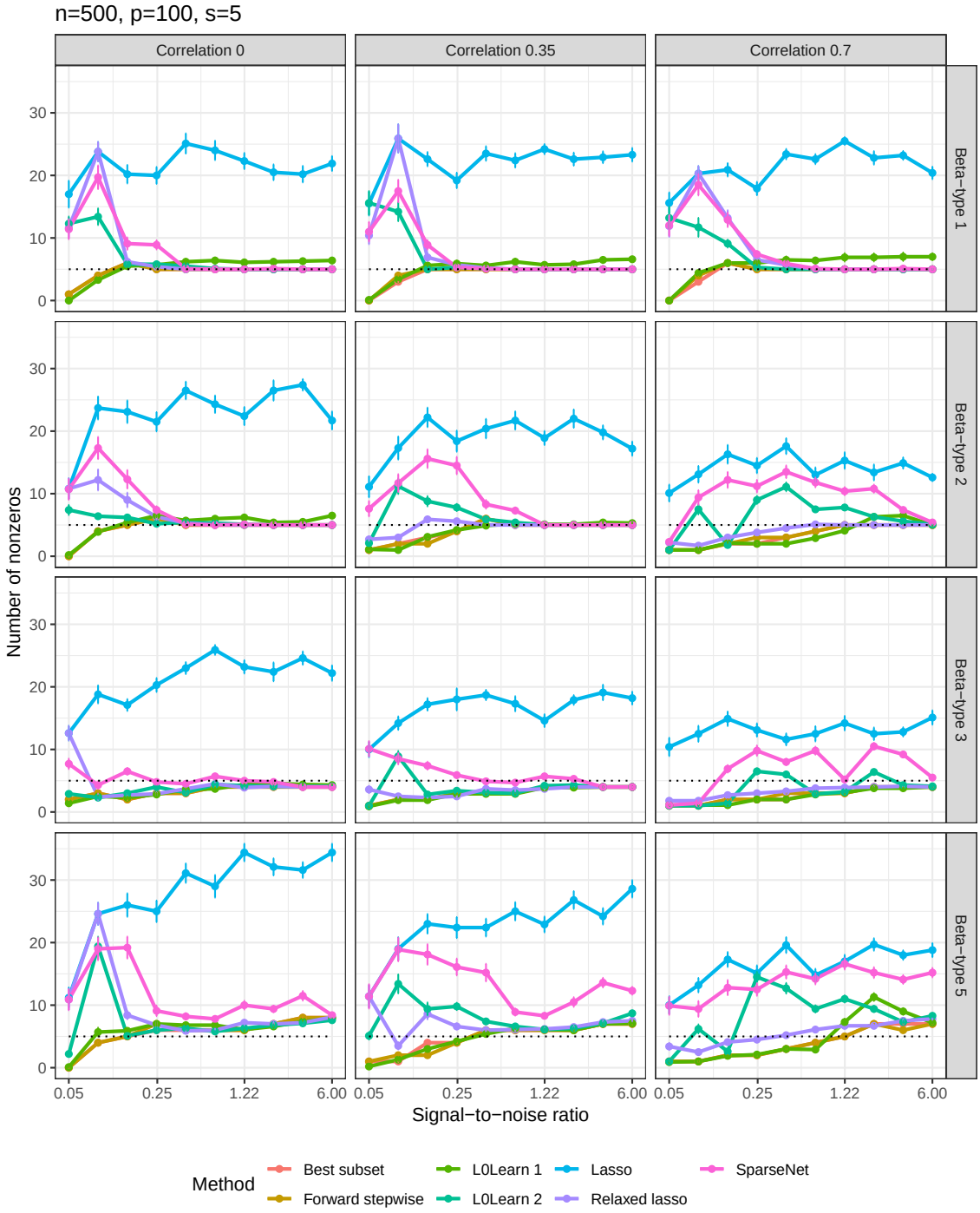
D.2.2 Relative test error (to Bayes)



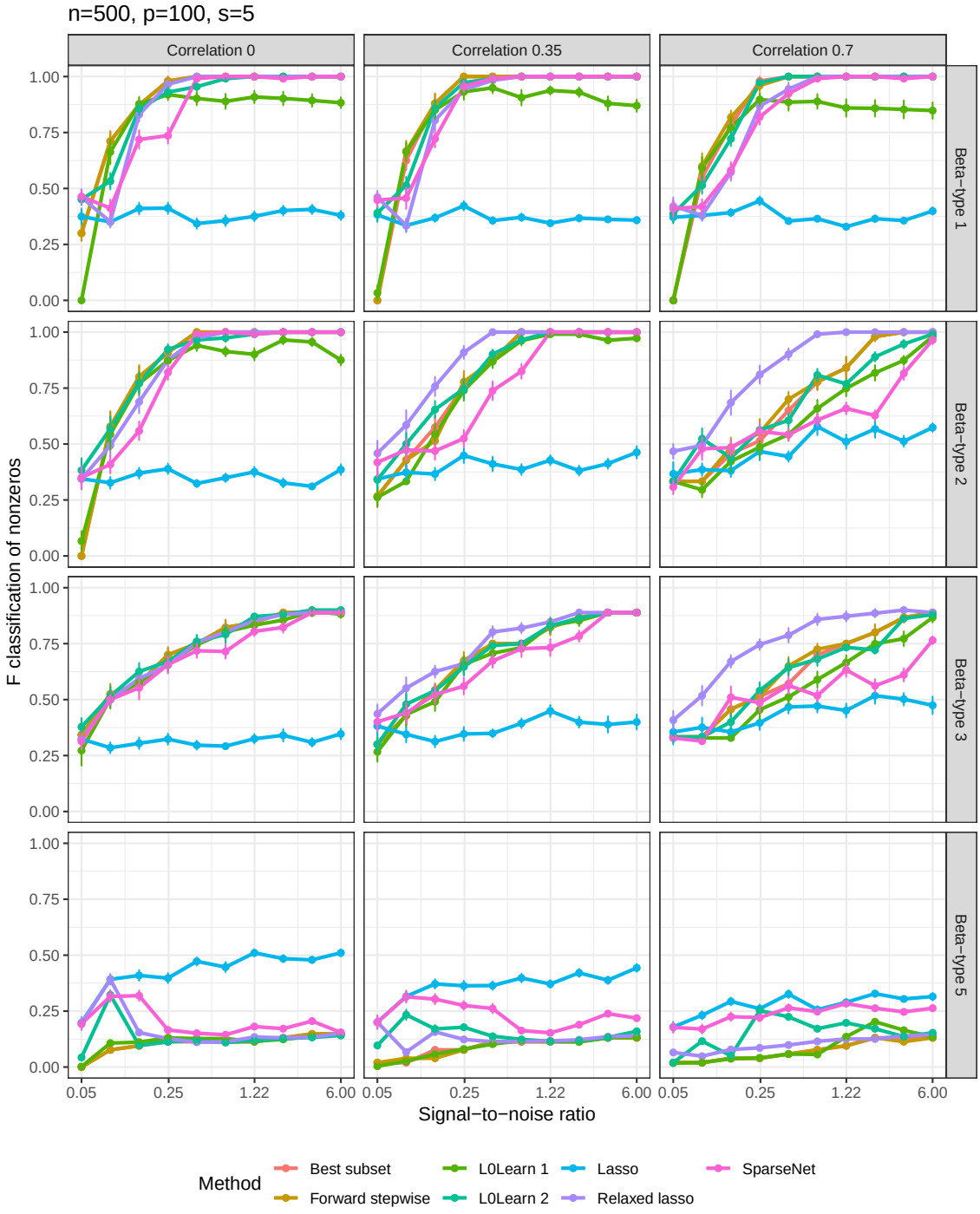
D.2.3 Proportion of variance explained



D.2.4 Number of nonzero coefficients

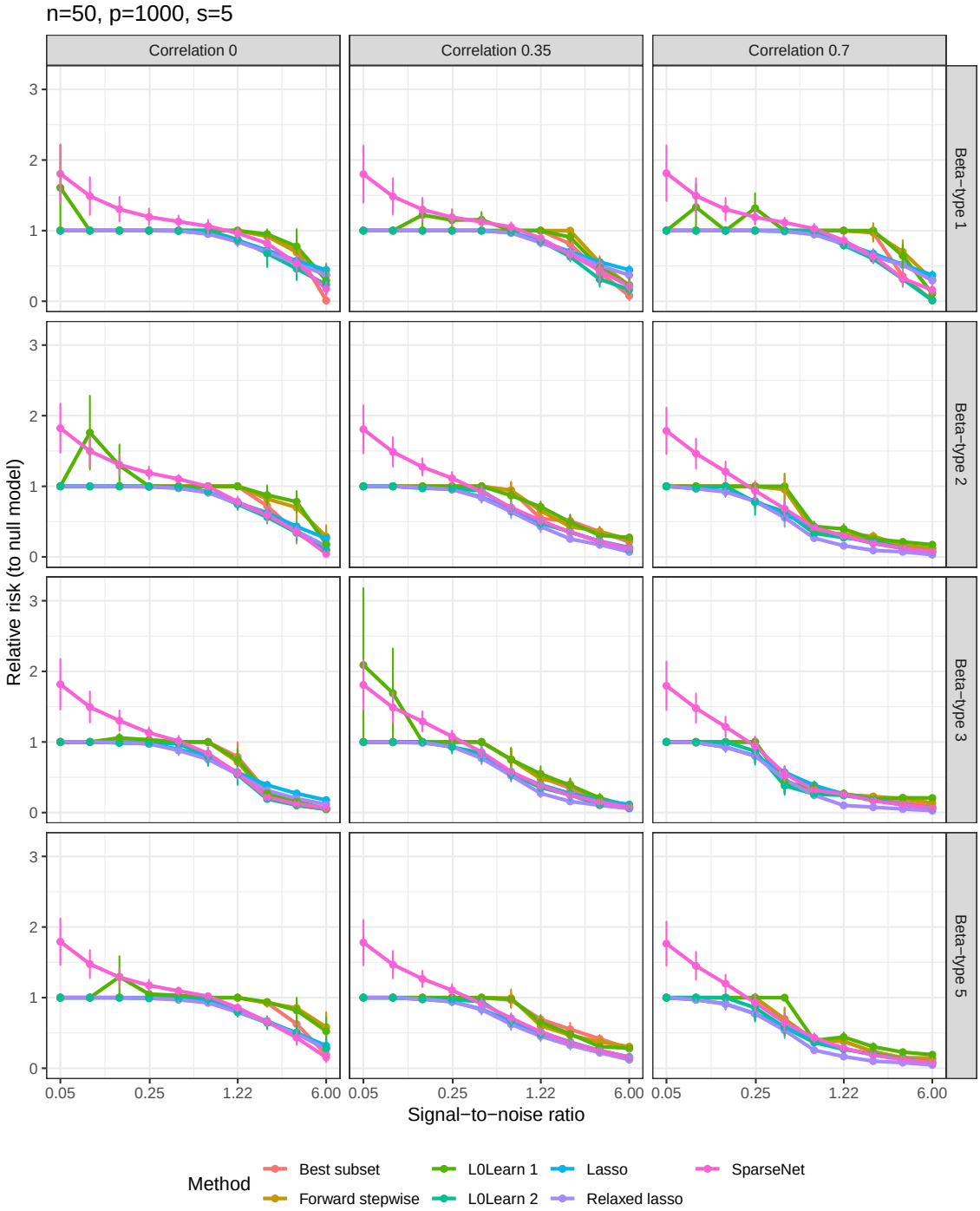


D.2.5 F-score

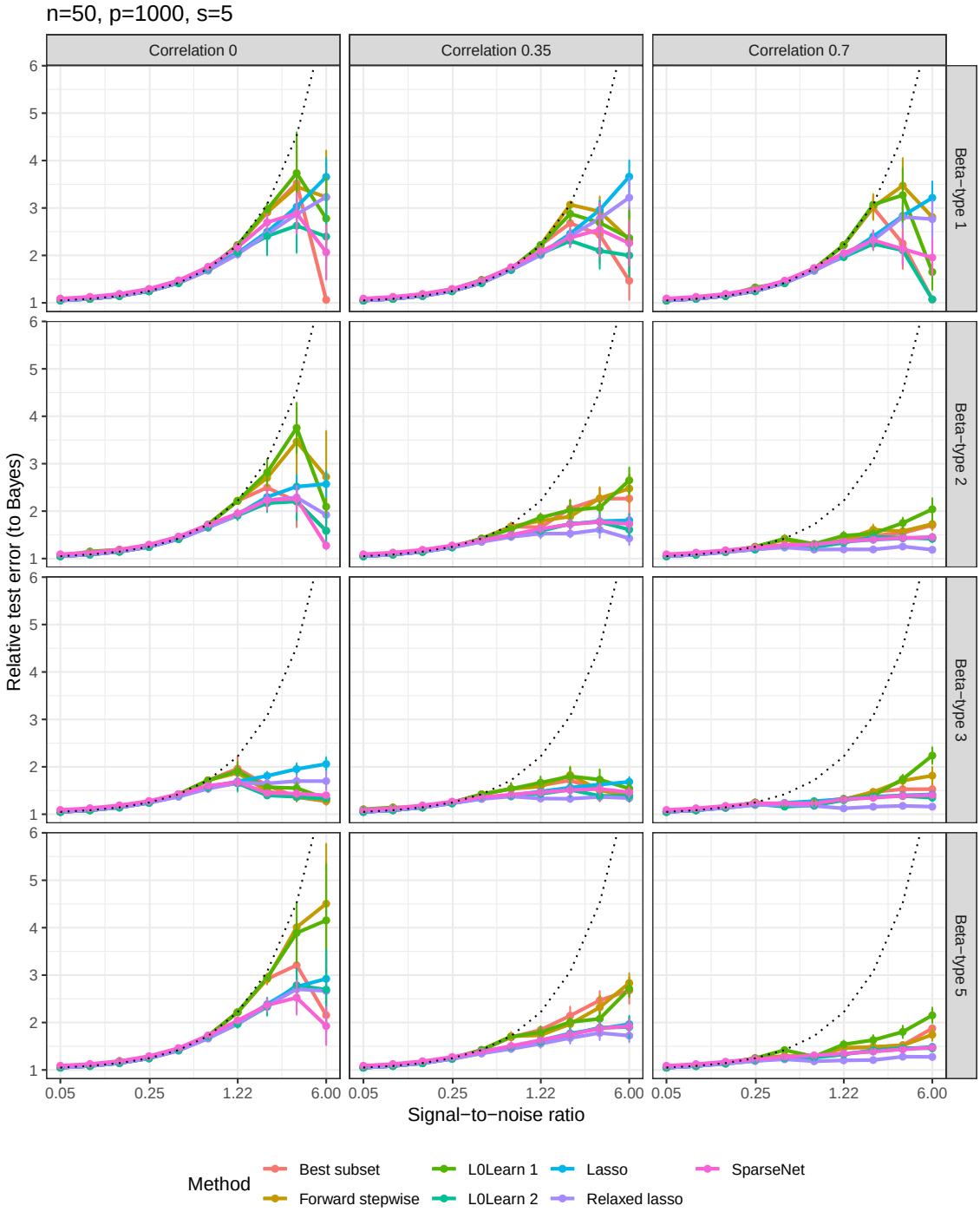

n=500, p=100, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

D.3 High-5 setting: $n = 50$, $p = 1000$, $s = 5$

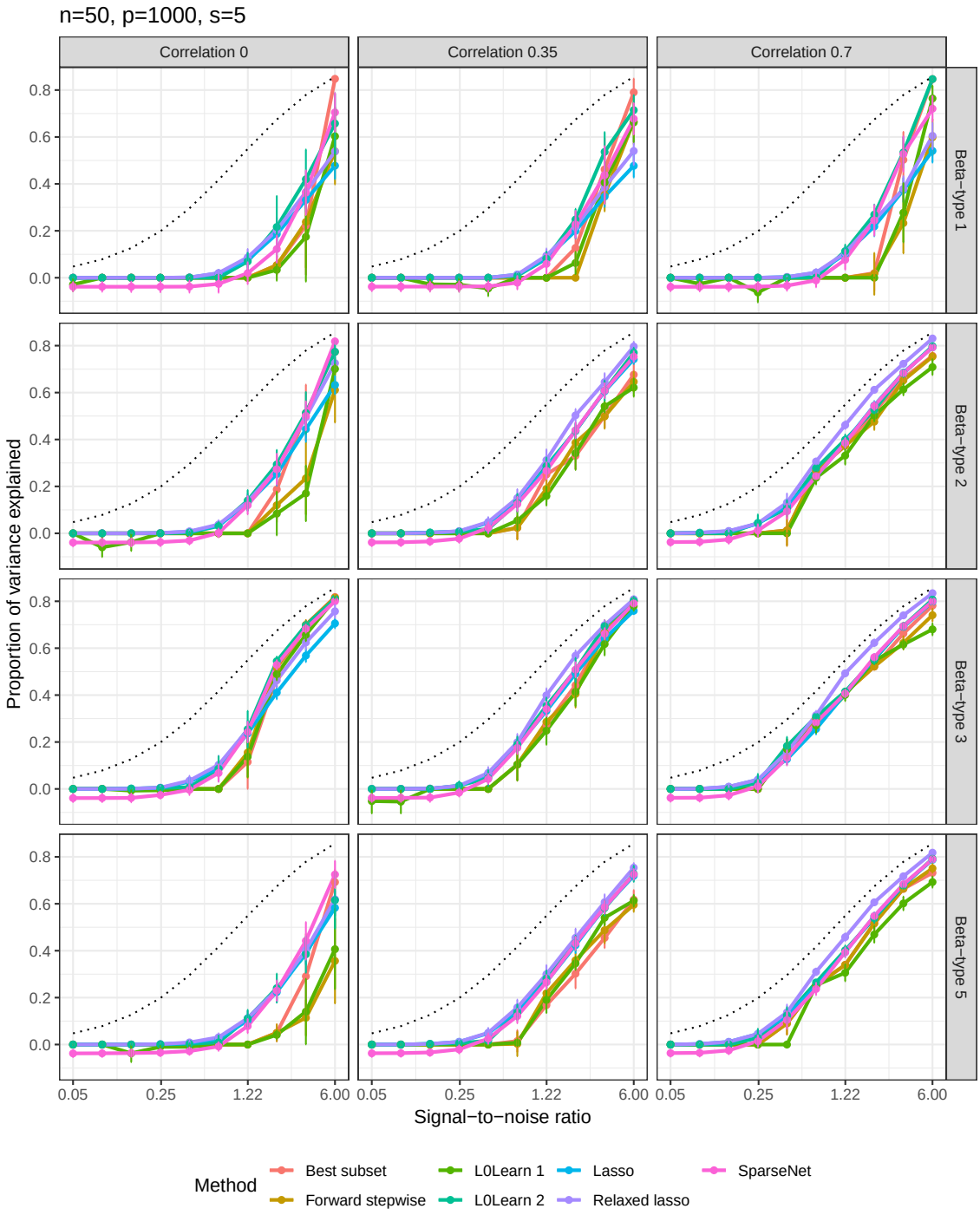
D.3.1 Relative risk (to null model)



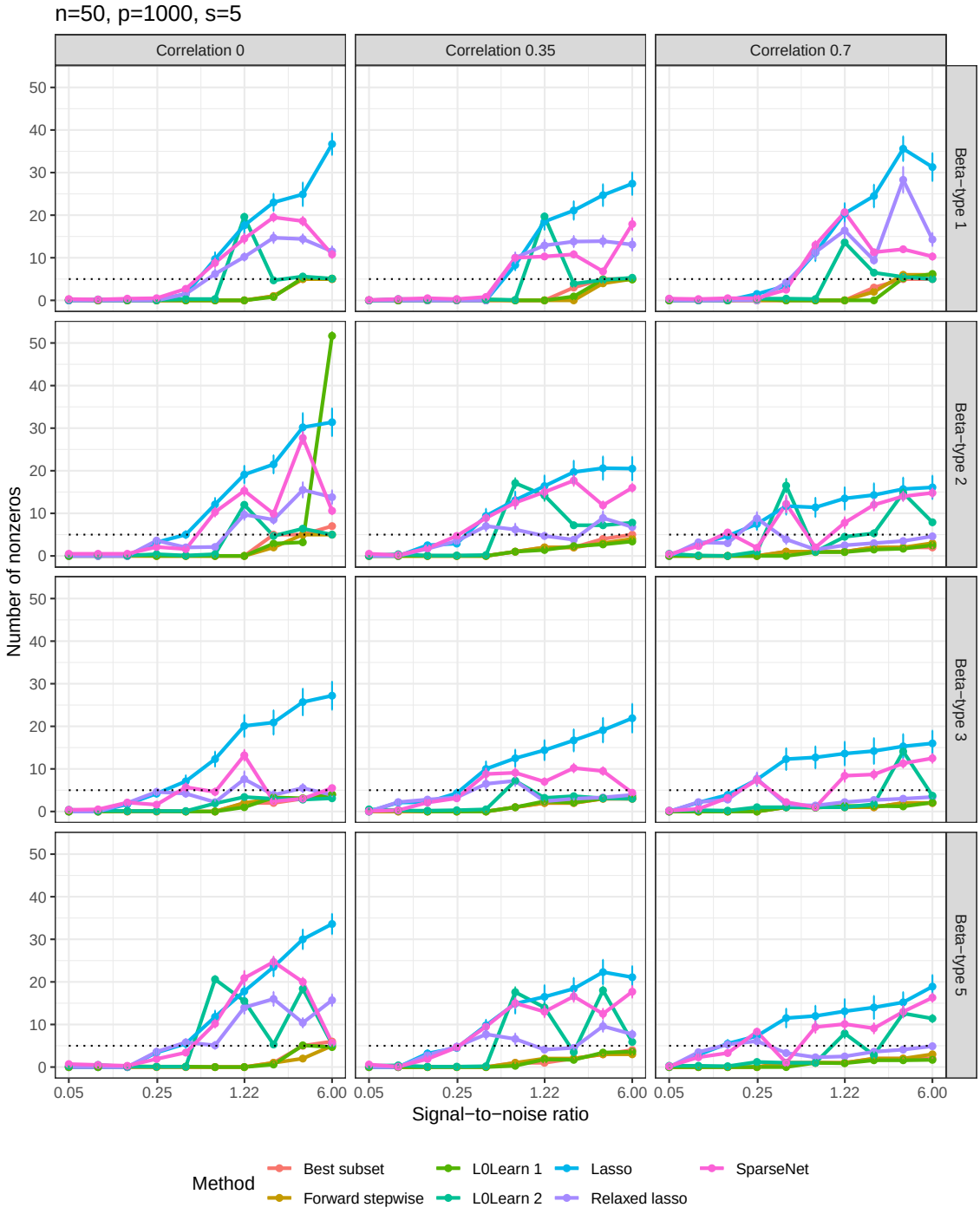
D.3.2 Relative test error (to Bayes)



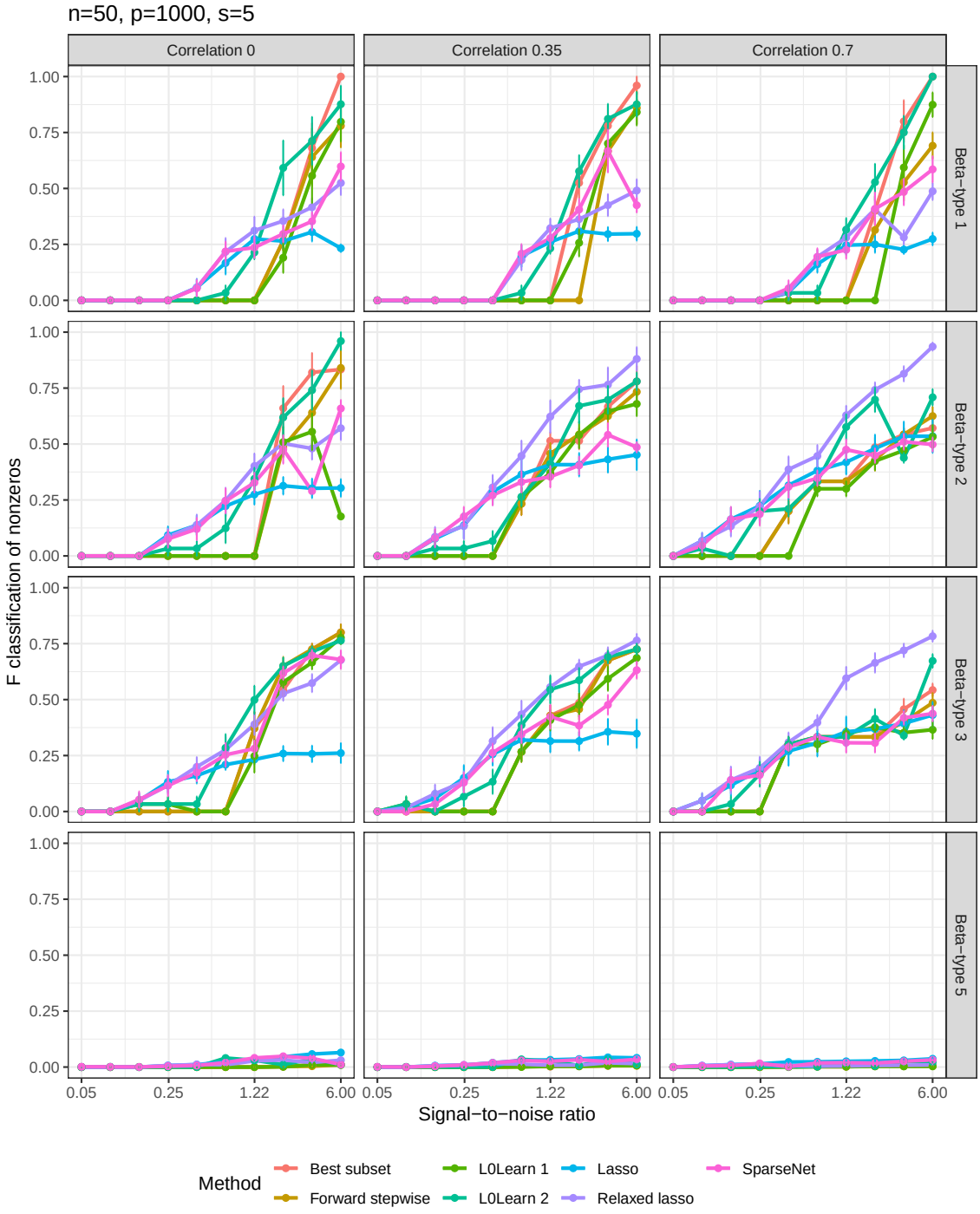
D.3.3 Proportion of variance explained



D.3.4 Number of nonzero coefficients

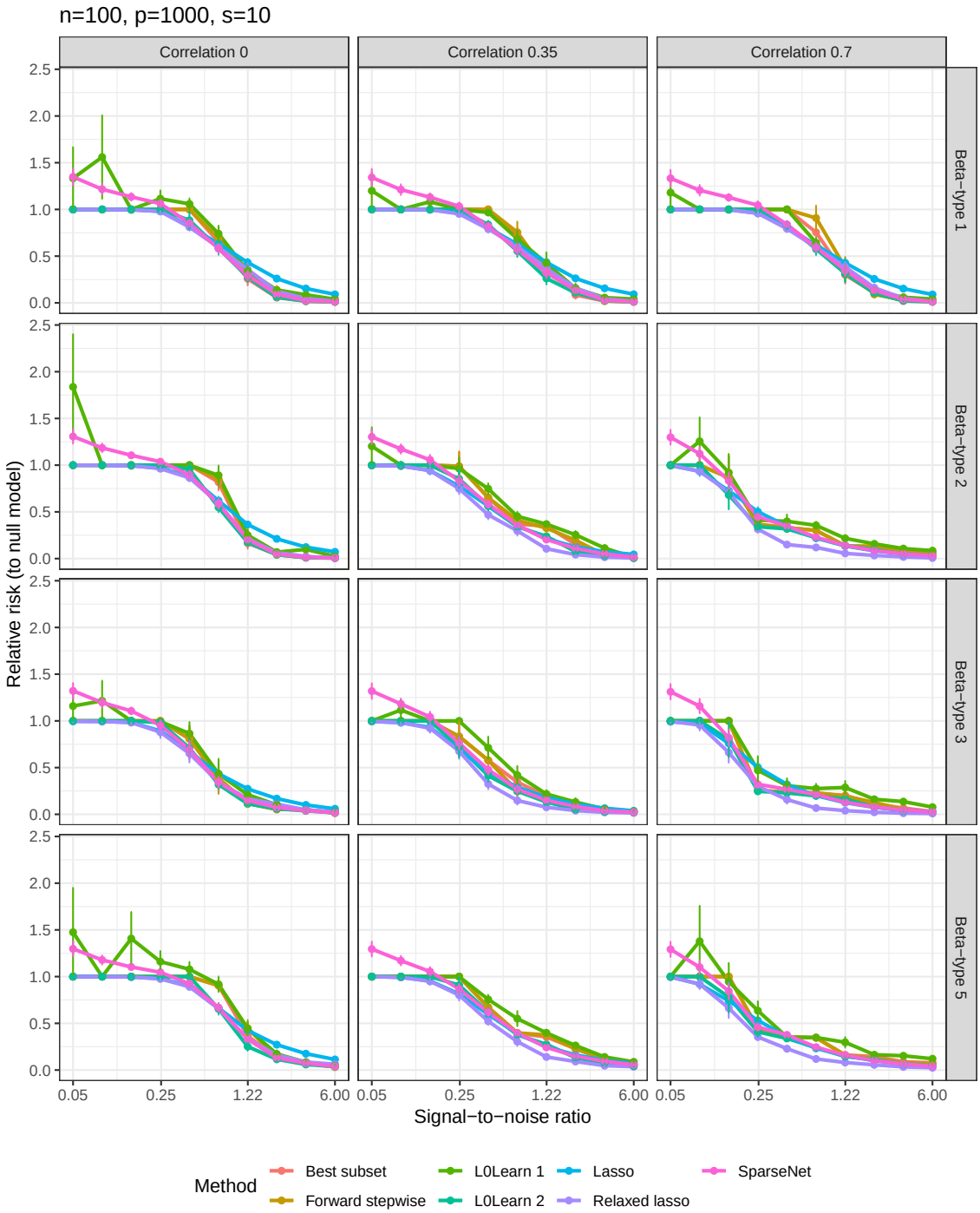


D.3.5 F-score

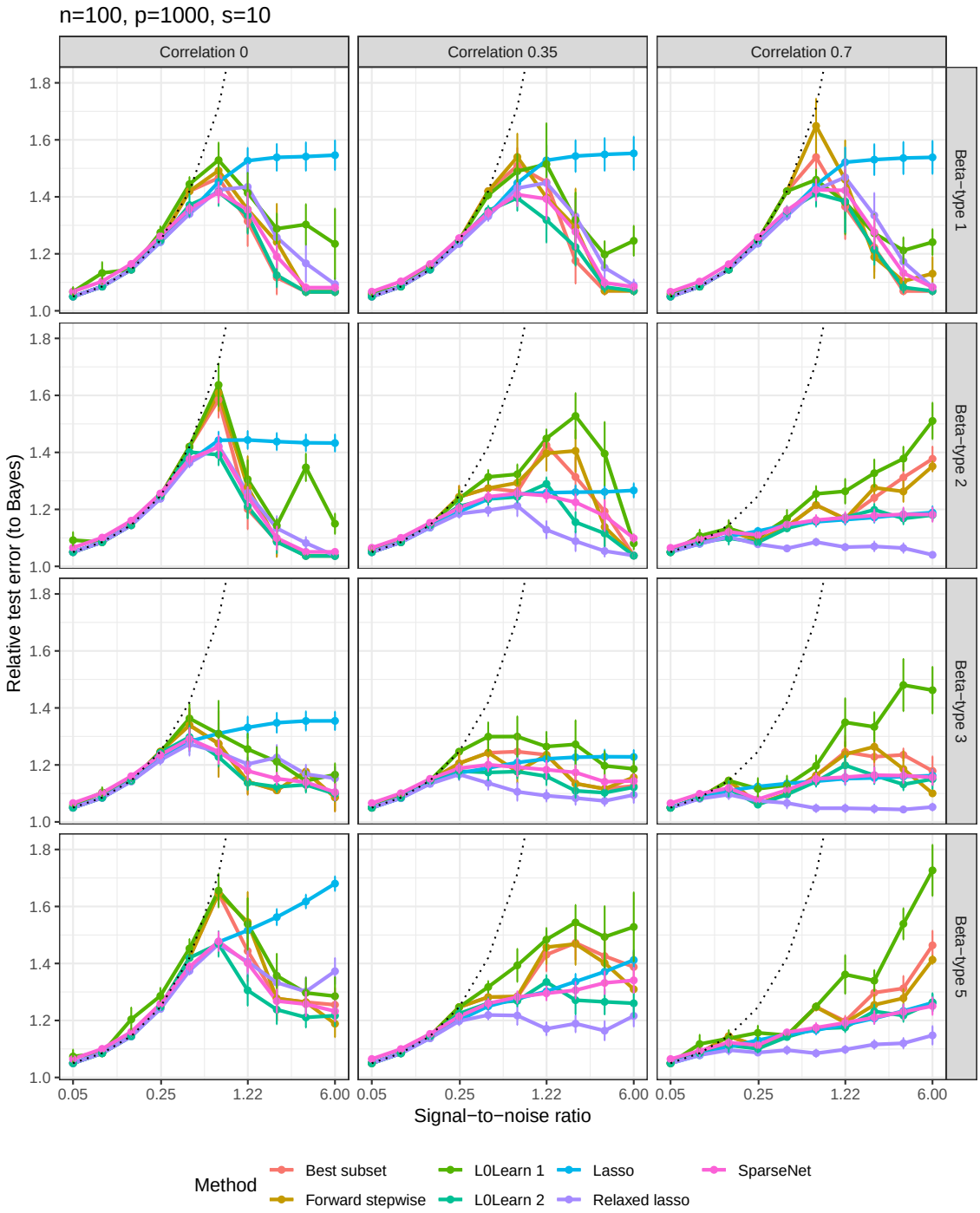

n=50, p=1000, s=5
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet

D.4 High-10 setting: $n = 100$, $p = 1000$, $s = 10$

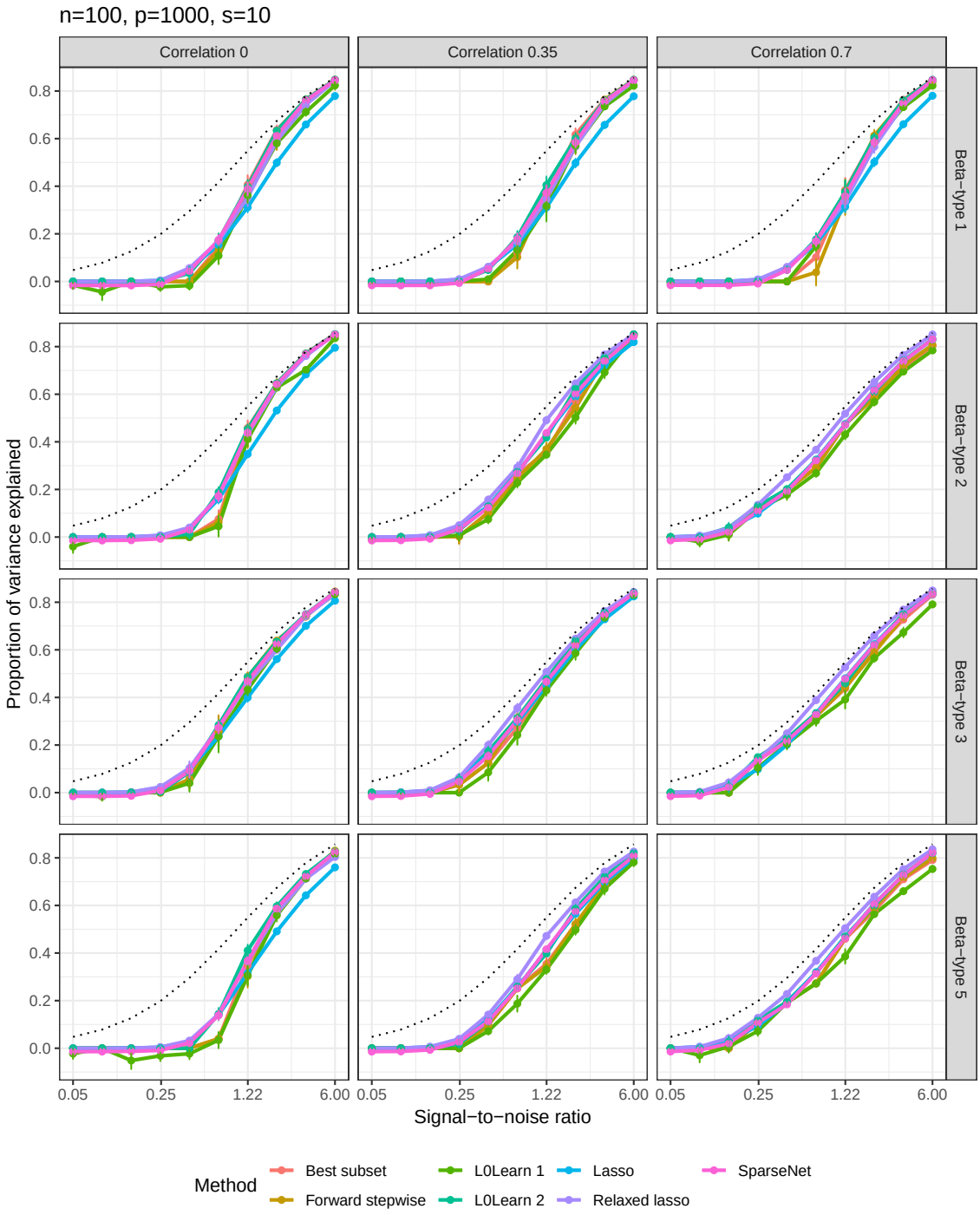
D.4.1 Relative risk (to null model)



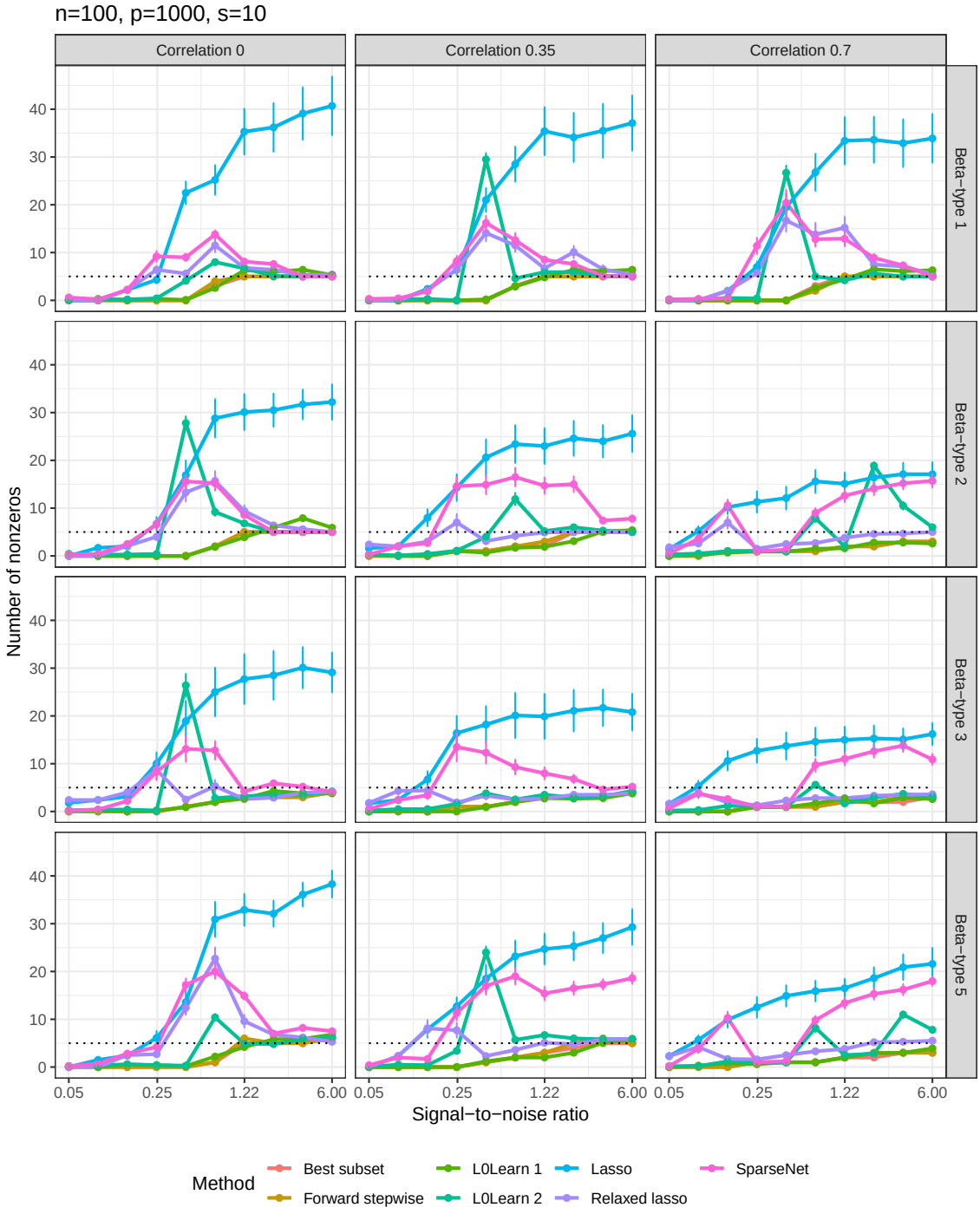
D.4.2 Relative test error (to Bayes)



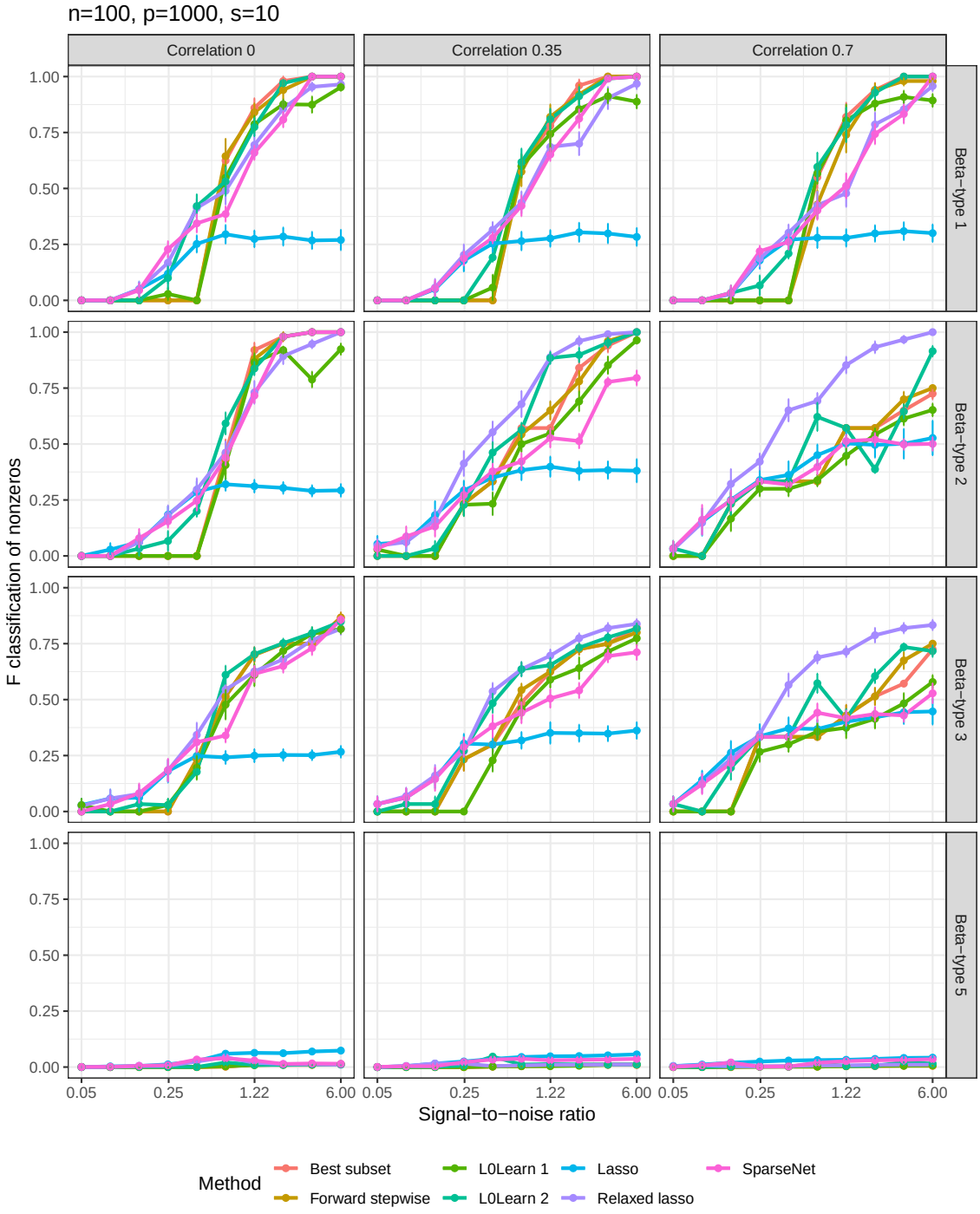
D.4.3 Proportion of variance explained



D.4.4 Number of nonzero coefficients



D.4.5 F-score


n=100, p=1000, s=10
Correlation 0
Correlation 0.35
Correlation 0.7
Beta−type 1
Beta−type 2
Beta−type 3
Beta−type 5
F classification of nonzeros
1.00
0.75
0.50
0.25
0.00
0.05
0.25
1.22
6.00
Signal−to−noise ratio
Method
Best subset
Forward stepwise
L0Learn 1
L0Learn 2
Lasso
Relaxed lasso
SparseNet